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MATHEMATICS

Topological Properties of Sets Admitting the Tschebycheff Systems

by

K. SIEKLUCKI

Presented by S. MAZUR on June 18, 1958

Let Z be a closed subset of an Euclidean space, or, more generally, let Z be a compact topological space. Let us denote by $C(Z)$ the set of all real and continuous functions defined on Z.

The system of functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in C(Z)$ is called the Tschebycheff system, if the equation $\sum_{k=1}^{n} c_k \varphi_k(p) = 0$ has at most $n-1$ roots for numbers c_k such that $\sum_{k=1}^{n} |c_k| > 0$. The importance of the notion of the Tschebycheff system for the theory of approximation lies in the fact that the following Haar's theorem [1] holds: the system of linear independent functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in C(Z)$ is the Tschebycheff system if, and only if, for each function $f \in C(Z)$ there exists exactly one linear form $\sum_{k=1}^{n} c_k \varphi_k$ such that

$$\sup_{p \in Z} \left| f(p) - \sum_{k=1}^{n} c_k \varphi_k(p) \right| = \inf_{c_1, \ldots, c_n} \sup_{p \in Z} \left| f(p) - \sum_{k=1}^{n} c_k \varphi_k(p) \right|.$$

Now, the following question arises: what is the topological structure of the space Z on which there exists at least one Tschebycheff system consisting of n functions ($n > 1$). S. Mazur expressed a conjecture that if such a system exists then Z is homeomorphic with a circle or with a subset of a segment. The purpose of this note is to prove this conjecture.

THEOREM. *If on the space Z there exists a Tschebycheff system of n functions, then, for even n, the space Z is homeomorphic with the subset of a segment, and, for odd n ($n > 1$), the space Z is homeomorphic with the circumference or with a subset of a segment.*

We can, of course, assume that the space Z contains an infinite set of points.

LEMMA. *If, on the space Z, there exists a Tschebycheff system of n functions ($n > 1$), then, for each open set $U \subset Z$ containing at least $n-2$ points, the set $Z-U$ is homeomorphic with the subset of a segment.*

Proof of the Lemma. Let us suppose that the system of functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in C(Z)$ is a Tschebycheff system. We define the mapping Φ of the space Z into the subset of the n-dimensional Euclidean space E^n in the following manner: $\Phi(p) = (\varphi_1(p), \varphi_2(p), \ldots \varphi_n(p))$. From the fact that $\varphi_1, \varphi_2, \ldots \varphi_n$ is a Tschebycheff system we immediately conclude that, if $q_1, q_2, \ldots, q_k$ are different points of Z ($k \leqslant n$), then the points $\Phi(q_1), \Phi(q_2), \ldots \Phi(q_k)$ are linearly independent. Let $p_1, p_2, \ldots, p_{n-2}$ be different points of the set U. Then there exists exactly one $(n-2)$-dimensional hyperplane passing through the origin of the co-ordinates x_0 and containing the points $\Phi(p_1), \Phi(p_2), \ldots, \Phi(p_{n-2})$. In the case of $n = 2$, the hyperplane H^{n-2} is reduced to the point x_0. Let $H^2 \subset E^n$ be a two-dimensional hyperplane passing through x_0 and perpendicular to the hyperplane H^{n-2}. Let us denote by S the circle with centre x_0 and radius 1, lying on the hyperplane H^2. If $x \in E^n$ then, by $g(x)$ we shall denote the orthogonal projection of the point x on the hyperplane H^2. If $x \in H^2$, $x \neq x_0$, then by $h(x)$ we shall denote the intersection point of S and the radius of S passing through x. If $p', p'' \in Z-U$, $p' \neq p''$, then $p_1, p_2, \ldots, p_{n-2}, p', p''$ are different points. Hence, the points $\Phi(p_1), \Phi(p_2), \ldots, \Phi(p_{n-2}), \Phi(p'), \Phi(p'')$ are linearly independent. This means, that the point $h\big(g(\Phi(p'))\big)$ is different from $h\big(g(\Phi(p''))\big)$ and from the point diametral to $h\big(g(\Phi(p''))\big)$ on S. Thus, we have proved that $y = h\big(g(\Phi(p))\big)$ defines the one to one correspondence of the set $Z-U$ into the proper subset of S. From the continuity of this correspondence, and from the fact that the set $Z-U$ is compact we conclude that it is a homeomorphism.

Proof of the theorem. Let us assume that the system of functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in C(Z)$ is the Tschebycheff system. We shall distinct three following cases.

Case I. The space Z is connected and it contains no subset homeomorphic with a circle. First, we shall prove that, for each point $p \in Z$, there exists a neighbourhood which is an arc without one or two endpoints. In order to prove this, let us choose the neighbourhood V of p in such a manner that the set $U = Z-\overline{V}$ should contain at least $n-2$ points. Then, by the lemma, there exists the homeomorphism h mapping the set $\overline{V} = Z-U$ into the subset of the segment. Let $\varepsilon > 0$ be such a number that the set W of all $q \in \overline{V}$ satisfying $|h(q)-h(p)| < \varepsilon$ is contained in V. The neighbourhood W has the required property, because, in the opposite case, there would exist such numbers $t', t'' \in h(W)$ and

point $q \in W$ that $h(p)-\varepsilon < t' < h(q) < t'' < h(p)+\varepsilon$. This means that, if Z_0 is a set of all $q \in W$ such that $t' < h(q) < t''$, then the sets Z_0, $Z-Z_0$ are non-void, closed, and disjoint, what is impossible because Z is a connected space. Now, let $p_0 \in Z$ be an arbitrary point such that some neighbourhood W of p_0 is the arc without the end points. Let us take the points $p_1, p_2 \in W$ such that p lies between p_1 and p_2, and let us denote by W_0 the set of all $p \in W$ lying between p_1 and p_2. The set $Z-W_0$, by the lemma, is homeomorphic with the subset of the segment. This means that the component Z_k of $Z-W_0$ containing the point p_k $(k=1,2)$ is homeomorphic with a connected subset of a segment, i. e. Z_k is an arc. Besides $Z_1 \cap Z_2 = 0$, because in the opposite case $Z_1 = Z_2$, and the set $W_0 \cup Z_1$ is homeomorphic with a circumference. But this is impossible because, by assumption, the space Z does not contain the subset homeomorphic with a circumference. Hence, we infer that the set $Z_1 \cup W_0 \cup Z_2$ is an arc. In order to complete the proof it is sufficient to see that $Z_1 \cup W_0 \cup Z_2 = Z$, because in the opposite case the sets $Z_1 \cup W_0 \cup Z_2$, $Z-(Z_1 \cup W_0 \cup Z_2)$ would be closed, non-void and disjoint, what is impossible, because Z is a connected space.

Case II. The space Z is not connected and does not contain the subset homeomorphic with a circumference. If there exists a decomposition $Z = Z_1 \cup Z_2$ such that Z_1, Z_2 are closed sets with void intersection and each of them contains at least $n-2$ points, then, by the lemma, each of the sets $Z_1 = Z-Z_2$, $Z_2 = Z-Z_1$ is homeomorphic with a subset of a segment, and the space Z has also the same property. In the opposite case let m be the most natural number, such that there exists the decomposition $Z = Z_1 \cup Z_2$, where Z_1, Z_2 are closed sets with void intersection, and the set Z_2 contains m points. Then the set Z_1 is connected. This means that Z_1 is a connected, compact topological space containing no subset homeomorphic with a circumference. Hence, Z_1 is homeomorphic with a segment, because the functions $\varphi_1, \varphi_2, \ldots \varphi_n$ considered on Z_1 also establish the Tschebycheff system. Hence, we obviously obtain that the space Z is homeomorphic with a subset of a segment.

Case III. The space Z contains a part Z_0 homeomorphic with a circumference. We shall prove that, in this case, the number n must be even and $Z = Z_0$. Let us denote by $p(t)$ the continuous function mapping the set of real numbers onto the set Z in such a manner that $p(t') = p(t'')$ if, and only if, $t'-t''$ is an integer.

By the definition of Tschebycheff's system we have

$$\Delta(t_1, t_2, \ldots, t_n) = \begin{vmatrix} \varphi_1(p(t_1)), \ldots, \varphi_n(p(t_1)) \\ \cdots\cdots\cdots\cdots\cdots \\ \varphi_1(p(t_n)), \ldots, \varphi_n(p(t_n)) \end{vmatrix} \neq 0,$$

for $t_1, t_2, \ldots, t_n$ such that the points $p(t_1), p(t_2), \ldots, p(t_n)$ are different. This property is, of course, satisfied in the particular case when

$0 \leqslant t_1 < t_2 < \ldots < t_n < 1$ or $0 < t_1 < t_2 < \ldots < t_n \leqslant 1$. We put $t'_n = (k-1)/k$ for $k = 1, 2, \ldots n$ and $t''_k = k/(k+1)$ for $k = 1, 2, \ldots, n-1$ and $t''_n = 1$.

Hence, we obtain

$$\Delta(t'_1 + s(t''_1 - t'_1), \ldots, t'_n + s(t''_n - t'_n)) \neq 0\,,$$

where $0 \leqslant s \leqslant 1$. Hence, by the continuity of the determinant, we have

$$\operatorname{sgn}\Delta(t'_1, t'_2, \ldots, t'_n) = \operatorname{sgn}\Delta(t''_1, t''_2, \ldots, t''_n)\,.$$

But it is obvious that

$$\Delta(t'_1, t'_2, \ldots, t'_n) = (-1)^{n-1} \cdot \Delta(t''_1, t''_2, \ldots, t''_n)\,,$$

so $(-1)^{n-1} = 1$. Hence, the number n is even. It is sufficient to show that $Z = Z_0$. Let us suppose that there exists the point $p_0 \in Z - Z_0$. Then, by the definition of Tschebycheff's system, we obtain

$$\Delta^*(t_1, t_2, \ldots t_{n-1}) = \begin{vmatrix} \varphi_1(p_0), \ldots, \varphi_n(p_0) \\ \varphi_1(p(t_1)), \ldots, \varphi_n(p(t_1)) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_1(p(t_{n-1})), \ldots, \varphi_n(p(t_{n-1})) \end{vmatrix} \neq 0$$

for $t_1, t_2, \ldots, t_{n-1}$ such that the points $p(t_1), p(t_2), \ldots, p(t_{n-1})$ are different. This property is also satisfied in the particular case when $0 \leqslant t_1 < t_2 < \ldots$ $\ldots < t_{n-1} = 1$ or $0 < t_1 < t_2 < \ldots < t_{n-1} \leqslant 1$. We put $t'_k = (k-1)/k$ for $k = 1, 2, \ldots, n-1$, $t''_k = k/(k+1)$ for $k = 1, 2, \ldots, n-2$ and $t''_{n-1} = 1$.

Hence, we have

$$\Delta^*(t'_1 + s(t''_1 - t'_1), \ldots, t'_{n-1} + s(t''_{n-1} - t'_{n-1})) \neq 0\,,$$

where $0 \leqslant s \leqslant 1$.

Hence, by the continuity of the determinant, we obtain

$$\operatorname{sgn}\Delta^*(t'_1, t'_2, \ldots, t'_{n-1}) = \operatorname{sgn}\Delta^*(t''_1, t''_2, \ldots, t''_{n-1})\,.$$

On the other hand, it is easily seen that

$$\Delta^*(t'_1, t'_2, \ldots t'_{n-1}) = (-1)^{n-2} \cdot \Delta^*(t''_1, t''_2, \ldots, t''_{n-1})\,,$$

so $(-1)^{n-2} = 1$. But this is impossible because the number n is even. Thus, we have proved that $Z = Z_0$.

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MATHEMATICS

On the Order of Points and Hyperplanes in n-dimensional Projective Geometry

by

L. DUBIKAJTIS

Presented by A. MOSTOWSKI on May 23, 1958

In one-dimensional projective geometry each triplet of points can be transformed into an arbitrary triplet of different points. It means that each ternary relation defined on the points and invariant under the projective transformation is either satisfied by all triplets of points or by none. This means that the ternary relation cannot be used to define the reciprocal situation of points.

It follows that the relation defining the order of points in one-dimensional projective geometry must have at least 4 arguments. The quadruplet relation of separation of two points by two is usually taken as such relation.

The same relation can also be applied in one, two, or generally n-dimensional geometry. In that case, such relation can be used to define the order of points on the line, and indirectly, on the whole space.

The question arises whether the reciprocal situation of points in the space would not be more suitably defined by a relation between a larger number of points placed in the whole space instead of by quadruplet relation between the four point on the line.

From the fact that, in n-dimensional projective geometry, each $n+2$ points (each $n+1$ of these do not lie on the hyperplane *)) can be transformed into arbitrary $n+2$ points with the same property, we obtain that the relation of "ordering" the points of n-dimensional space must have at least $n+3$ arguments.

In this paper we give the definition of such relation (with $n+3$ arguments) which permits to define the reciprocal situation of points in n-dimensional space. So defined relation is a generalization of the

*) In n-dimensional geometry we understand by hyperplane the $(n-1)$-dimensional plane.

relation ordering the points on the line, and has a certain number of analogous properties.

Let us consider the order of points on the projective line. Usually the relation of separation of two points by two is used to define this order. But it can be replaced by the following relation:

Definition 1. We shall say that the points A, B, C, D form a cycle A, C, B, D instead of saying that the points A, B, C, D lie on the same line and the pairs A, B and C, D are separated. This fact will be denoted by $\{A, C, B, D\}$.

Of course, so defined relation is equivalent to the relation of separation and can be used to define the order of points on the line.

The above relation has the following properties.

THEOREM 1. *If* $\{A_1, A_2, A_3, A_4\}$, *then* A_1, A_2, A_3, A_4 *are different points lying on the line.*

THEOREM 2. *If the points* A_1, A_2, A_3, A_4 *are different, and if they lie on the same line, then there exists such permutation* i_1, i_2, i_3, i_4 *of integers* $1, 2, 3, 4$ *that* $\{A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}\}$.

THEOREM 3. *If* $\{A_1, A_2, A_3, A_4\}$ *then* $\{A_2, A_3, A_4, A_1\}$ *and* $\{A_4, A_3, A_2, A_1\}$.

The following logical equivalence can be easily obtained from this theorem: $\{A_1, A_2, A_3, A_4\} \equiv \{A_2, A_3, A_4, A_1\} \equiv \{A_3, A_4, A_1, A_2\} \equiv \{A_4, A_1, A_2, A_3\} \equiv \{A_4, A_3, A_2, A_1\} \equiv \{A_3, A_2, A_1, A_4\} \equiv \{A_2, A_1, A_4, A_3\} \equiv \equiv \{A_1, A_4, A_3, A_2\}$.

THEOREM 4. *The condition* $\{A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}\}$ *cannot be satisfied by more than* 8 (*out of* 24 *possible*) *permutations* i_1, i_2, i_3, i_4 *of integers* $1, 2, 3, 4$.

Below we give the definition of the cycle of $n+3$ points. This cycle will be denoted by $\{A_1, A_2, \ldots, A_{n+2}, A_{n+3}\}$ and will satisfy the following theorems analogous to 1—4.

THEOREM 1′. *If* $\{A_1, A_2, \ldots, A_{n+3}\}$ *then the points* $A_1, A_2, \ldots, A_{n+3}$ *are contained in* n*-dimensional space and each* $n+1$ *of them do not lie in the same hyperplane.*

THEOREM 2′. *If the points* $A_1, A_2, \ldots, A_{n+3}$ *are contained in* n*-dimensional space and each* $n+1$ *of them do not lie on the same hyperplane, then there exists such permutation* $i_1, i_2, \ldots, i_{n+3}$ *of integers* $1, 2, \ldots, n+3$ *that* $\{A_{i_1}, A_{i_2}, \ldots, A_{i_{n+3}}\}$.

THEOREM 3′. *If* $\{A_1, A_2, \ldots, A_{n+2}, A_{n+3}\}$, *then* $\{A_{n+3}, A_1, A_2, \ldots, A_{n+2}\}$ *and* $\{A_{n+3}, A_{n+2}, \ldots, A_2, A_1\}$.

THEOREM 4′. *The condition* $\{A_{i_1}, A_{i_2}, \ldots, A_{i_{n+2}}, A_{i_{n+3}}\}$ *cannot be satisfied by more than* $2(n+3)$ (*out of* $n!$ *possible*) *permutations of integers* $1, 2, \ldots, n+3$.

From the last two theorems it follows that the cycle $\{A_1, A_2, \ldots, A_{n+3}\}$ is equivalent to exactly $2(n+3)$ cycles of the form $\{A_{i_1}, A_{i_2}, \ldots, A_{i_{n+3}}\}$ (by Theorem 3′). This cycle is contradictory with every other cycle of such form.

Before the definition of the cycle of $n+3$ points satisfying the above theorems, we shall define the cycle of four hyperplanes. It is the dual notion to the notion of cycle of four points given by Definition 1.

Definition 2. We shall say that the hyperplanes a, b, c, d form the cycle a, b, c, d (this means, that $\{a, b, c, d\}$) if, and only if, these hyperplanes form the pencil (i. e. they pass through some $(k-2)$-dimensional plane, which is called the edge of the pencil), and the corresponding intersection points A, B, C, D assigned by the line not passing through the edge of pencil form a cycle A, B, C, D.

Definition 3. We shall say that the points $A_1, \ldots, A_{n+3}$ form the cycle $A_1, \ldots, A_{n+3}$, i. e. $\{A_1, \ldots, A_{n+3}\}$ if, and only if, for each four-integer p, q, r, s satisfying the condition $1 \leqslant p < q < r < s \leqslant n+3$, we have $\{a_p, a_q, a_r, a_s\}$ (where a_j denotes the hyperplane generated by n points $A_1, \ldots, A_{p-1}, A_{p+1}, \ldots, A_{q-1}, A_{q+1}, \ldots, A_{r-1}, A_{r+1}, \ldots, A_{s-1}, A_{s+1}, \ldots, A_{n+3}, A_j$).

It can be proved, that so defined cycle of $n+3$ points satisfies Theorems 1′-4′.

The above definition holds for each integer $n \geqslant 2$. In particular we may speak of a cycle of 5 points on a 2-dimensional plane, and of the cycle of 6 points on a 3-dimensional space.

In n-dimensional space we can also define the dual notion relatively to the notion of the cycle of $n+3$ points, i. e. the notion of the cycle of $n+3$ hyperplanes. It is obvious, that this notion satisfies the dual theorems relatively to Theorems 1′-4′.

This notion is defined for $n = 2, 3, \ldots$ because its definition is dual relatively to Definition 3. Thus, we may speak of a cycle of five lines on the two-dimensional plane and of a cycle of six two-dimensional planes contained in a three-dimensional space etc. Generally, we may speak of a cycle of $k+3$ $(k-1)$-dimensional planes contained in k-dimensional space $(k = 2, 3, \ldots)$.

This notion can also be extended to the case of $n = 1$. If we shall consider the points A, B, C, D lying on the line as zero-dimensional planes contained in one-dimensional plane, then we can assume that these planes form a cycle A, B, C, D, if, and only if, they satisfy the conditions of Definition 1.

Thus extended, the notion of the cycle of $k+3$ $(k-1)$-dimensional planes satisfies the following theorem:

Theorem 5. *Let us denote by b_{ij} the plane obtained by intersection of the planes a_i and a_j. Then, in n-dimensional space the condition $\{a_1, \ldots, a_{k+3}\}$*

(where $2 \leqslant k \leqslant n$) is satisfied if, and only if, for each $j = 1, 2, \ldots, k+3$ a_j are $(k-1)$-dimensional planes, $b_{j,i}$ (for each $i \neq j$) are $(k-2)$-dimensional planes and $\{b_{j,1}, \ldots, b_{j,j-1}, b_{j,j+1}, \ldots, b_{j,k+3}\}$.

This theorem permits to define, by induction, the cycle of $k+3$ planes.

The above theorem indicates a kind of a heredity of the cycle of planes. Namely, if the planes $a_1, \ldots, a_{k+3}$ form a cycle, then on each plane a_i (where $i = 1, \ldots, k+3$), the planes $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+3}$, when cutting across it, generate a cycle of planes smaller by one dimension, preserving the same successive order.

The following problem can be formulated in connection with the fact that starting from the cycle of $n+3$ points (or $n+3$ hyperplanes), we can define the notion of separation of two points by two.

PROBLEM 1. *Which properties of the cycle of $n+3$ points (or hyperplanes) ought to be taken as axioms of n-dimensional projective geometry in order to obtain from these axioms all classical axioms of the order of this geometry (defining the relation of separation of two points by two)?*

It is interesting to observe that the cycle of 5 lines (on two-dimensional plane) has very simple geometrical interpretation. Namely, 5 lines (each three of them do not have a common point) divide the projective plane into 11 regions. Among them are 5 regions bounded by 3 lines, 5 — bounded by 4 lines, and only one — by 5 lines. It is easy to verify that the succesion of lines bounding the last region is exactly such as the succesion of lines in the cycle generated by them.

The above interpretation permits to see from a diagram which cycle forms the given lines.

PROBLEM 2. *Find a similar geometrical interpretation of the cycle of $n+3$ hyperplanes in n-dimensional space.*

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
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МАТЕМАТИКА

Б. В. ГНЕДЕНКО

О КРИТЕРИИ ВИЛЬКОКСОНА СРАВНЕНИЯ ДВУХ ВЫБОРОК

Представлено Э. МАРЧЕВСКИМ 5 июля 1958

1. Одна из важных задач математической статистики, часто возникающая в научной и технической практике, состоит в следующем:

Имеются результаты двух серий независимых измерений

$$x_1, x_2, \ldots, x_m,$$

$$y_1, y_2, \ldots, y_n$$

соответственно случайных величин ξ и η. На основании этих данных требуется подтвердить или опровергнуть гипотезу, согласно которой обе последовательности чисел получены в результате наблюдений над случайными величинами с одним и тем же распределением вероятностей.

Имеется довольно большое число подходов к решению поставленной задачи, каждый из которых приводит к определенному статистическому правилу, позволяющему дать оценку вероятности совпадения исходных распределений. Среди таких критериев большую известность приобрел критерий, предложенный в 1945 г. Ф. Вилькоксоном [1]. Критерий Вилькоксона нашел широкое практическое применение, а также очень высокую оценку. Так, в недавней книге Сиднея Сигеля ([2], стр. 75) критерий Вилькоксона характеризуется как „наиболее полезный критерий для наук о поведении". В связи с этим обстоятельством, за последние годы этому критерию был посвящен ряд интересных теоретических исследований.

2. Пусть функция распределения величины ξ равна $F_1(x)$, а величины $\eta - F_2(x)$. О функциях $F_1(x)$ и $F_2(x)$ предполагается, что они непрерывны (или, по меньшей мере, не имеют общих точек разрыва). Гипотеза H_0, подлежащая проверке, состоит в том, что при всех x

$$F_1(x) = F_2(x)$$

Вилькоксон предложил рассматривать статистику U, равную числу пар (i, j) $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, для которых $y_j < x_i$. Таким образом, по

определению

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon(x_i - y_j),$$

где

$$\varepsilon(z) = \begin{cases} 0, & \text{если } z < 0, \\ 1, & \text{если } z \geqslant 0. \end{cases}$$

Интуитивная идея, положенная в основу рассмотрения статистики U состоит в том, что как слишком малое, так и слишком большое значение U, должно свидетельствовать о малой вероятности гипотезы H_0. Если гипотеза H_0 верна, то должно быть достаточно хорошее перемешивание величин x_i и y_j, в результате которого U примет значение близкое к своему математическому ожиданию. В то же время, если $F_1(x)$ и $F_2(x)$ различны и сосредоточены на двух непересекающихся отрезках, то, в зависимости от того, какому отрезку будет принадлежать каждое из этих распределений, число U будет равно или минимальному значению 0 или максимальному — mn.

3. Легко подсчитать, что если гипотеза H_0 верна, то

$$MU = \frac{mn}{2},$$

$$DU = \frac{1}{12} mn(m+n+1).$$

Манн и Уитней [3] доказали, что распределение U асимптотически нормально. Точнее, они показали, что

$$\lim_{m,n\to\infty} P\left\{\frac{U - \frac{mn}{2}}{\sqrt{DU}} < z\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt.$$

Согласно критерию Вилькоксона, гипотеза H_0 принимается, если

$$\left|U - \frac{mn}{2}\right|$$

невелика и отбрасывается в ином случае.

Имеются таблицы, позволяющие использовать критерий Вилькоксона на практике.

4. В настоящей заметке указывается пример, когда заведомо $F_1(x)$ отлично от $F_2(x,)$, но критерий Вилькоксона при увеличении числа испытаний с вероятностью, стремящейся к 1, будет приводить к утверждению правильности равенства $F_1(x) \equiv F_2(x)$.

Пусть (a, b), (c, d), (e, f) — три непересекающихся отрезка, причем $b \leqslant c$, $d \leqslant e$. Пусть $F_1(x)$ и $F_2(x)$ таковы, что

$$F_1(d) - F_1(c) = 1\,,$$

$$F_2(b) - F_2(a) = F_2(f) - F_2(e) = \tfrac{1}{2}\,.$$

Очевидно, что при больших n приблизительно половина наблюденных величин y_j попадет в отрезок (a, b). Легко подсчитать, что

$$MU = \frac{mn}{2}$$

и

$$DU = \tfrac{1}{4}mn\,.$$

Таким образом, в условиях примера, применение критерия Вилькоксона будет приводить нас, вопреки истине, к убеждению в правильности гипотезы H_0: $F_1(x) = F_2(x)$. Вероятность того, что в результате процедуры Вилькоксона гипотеза H_0 будет отброшена, стремится к нулю с возрастанием n.

5. Построенный пример наводит на мысль о необходимости введения понятия „состоятельности критерия". Критерий называется состоятельным, если различает любые два распределения с вероятностью, стремящейся к единице при увеличении числа наблюдений. В этом смысле критерий Вилькоксона является несостоятельным. В то же время легко заметить, что критерий Колмогорова-Смирнова состоятелен.

Одновременно необходимо отметить, что оценки мощности критериев при рассмотрении близких альтернатив могут привести к выводам, которые при практических применениях приводят к ложным результатам. Известно (см. напр., [4]), что критерий Вилькоксона в некотором смысле имеет максимальную мощность среди ряда других критериев (в том числе и критерия Колмогорова-Смирнова), а именно $3/\pi = 0{,}995\ldots$ Приведенный выше пример показывает, что если в практических наблюдениях критерий Вилькоксона дает совпадение распределений, то к этому заключению следует относиться с сомнением.

ИНСТИТУТ МАТЕМАТИКИ АКАДЕМИИ НАУК УССР

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

[1] F. Wilcoxon, *Individual comparisons by ranking methods*, Biometrics Bull., **1** (1945), 80-83.

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B. V. GNEDENKO, ON THE WILCOXON TEST OF COMPARING OF TWO SAMPLES . . . . . .

A statistical test will be called *consistent* if it distinguishes two different probability distributions with the probability tending to 1, when the number of experiments increases to infinity. It is easy to verify that the Kolmogorov-Smirnov test is consistent.

The paper contains an example proving that *the Wilcoxon test is non consistent.*

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MATHÉMATIQUE

Sur les opérations transformant séries convergentes en séries convergentes

par

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Présenté par T. WAŻEWSKI le 13 août, 1958

Nous donnons une condition nécessaire et suffisante pour qu'une opération $y = f(x)$ transforme chaque série convergente, dans un espace de Banach, en série convergente. Notre théorème fournit, en même temps, une condition nécessaire et suffisante pour qu'une opération $f(x)$ soit linéaire (et continue) dans un voisinage de $x = 0$.

HYPOTHÈSE PRÉLIMINAIRE H: *Soient X et Y deux espaces linéaires et complets de Banach à norme homogène.*

Nous n'envisageons dans la suite que les fonctions (dites aussi opérations) $f(x), F(x), \ldots$ qui 1°) sont définies dans l'espace X *tout entier* et 2°) dont les valeurs appartiennent à l'espace Y.

Transformation des séries au moyen d'une opération. Soit

$$y = f(x) \tag{1}$$

une fonction et

$$\sum_{j=1}^{\infty} x_j \tag{2}$$

une série infinie (convergente ou divergente). La série

$$\sum_{j=1}^{\infty} f(x_j) \tag{3}$$

(convergente ou non) constitue, par définition, l'image de la série (2) par l'intermédiaire de (1). On dira que f transforme la série (2) en série (3). Dans les considérations qui suivent interviendront les propriétés suivantes relatives à $f(x)$.

PROPRIÉTÉ A, qui consiste en ce que $f(x)$ *transforme chaque série convergente en série convergente.*

Propriété B, qui consiste en ce que 1°) f *est continue pour* $x = 0$ et que 2°) *il existe un* r $(0 < r < +\infty)$ *tel que* $f(a)+f(b) = f(a+b)$ *lorsque* $a \in X$, $b \in X$, $|a| \leqslant r$, $|b| \leqslant r$.

Propriété C, qui consiste en ce qu'*il existe une opération linéaire* *) $F(x)$ *et un* r $(0 < r < +\infty)$, *tels que*

$$f(x) = F(x) \quad \textit{lorsque} \quad |x| \leqslant r\,.$$

Cette propriété exprime que l'opération $f(x)$ „est localement linéaire" c'est-à-dire linéaire dans un voisinage de $x = 0$.

Remarque 1. On sait, que toute opération linéaire jouit de la propriété A.

Théorème 1. *Dans l'hypothèse préliminaire* H *les propriétés* A, B *et* C *sont équivalentes entre elles.*

Démonstration. On a évidemment $C \supset B$. En admettant B et en posant $F(0) = 0$, $F(x) = \frac{|x|}{r} f\left(\frac{r}{|x|} \cdot x\right)$ pour $x \neq 0$ on prouve facilement, que $F(x)$ est linéaire et jouit de la propriété C *). On a donc $B \supset C$, ce qui conduit à l'équivalence $B \equiv C$. En observant que les séries

$$\sum_{j=1}^{\infty} f(x_j) \quad \text{et} \quad \sum_{j=p}^{\infty} f(x_j) \quad (\text{où } p \geqslant 1)$$

ne peuvent être convergentes que simultanément et en s'appuyant sur la Remarque 1, on démontre aisément que $C \supset A$. Pour terminer la démonstration il suffit de prouver que $A \supset B$ (car on sait déjà que $B \equiv C$). À cet effet il suffit évidemment de prouver le suivant

Lemme 1. *Si l'opération* $f(x)$ *jouit de la propriété* A *alors elle possède les propriétés suivantes*

Propriété P_1. *Il existe un* r $(0 < r < +\infty)$ *tel que les hypothèses*

$$q_i \in X, \quad |q^i| \leqslant r, \quad (i = 1, 2, 3), \quad q_1 + q_2 + q_3 = 0$$

entraînent la relation

$$f(q_1) + f(q_2) + f(q_3) = 0\,.$$

Propriété P_2. $f(0) = 0$.

Propriété P_3. $f(-x) = -f(x)$ *lorsque* $|x| \leqslant r$.

Propriété P_4. $f(a+b) = f(a) + f(b)$ *lorsque* $a \in X$, $B \in X$, $|a| \leqslant r$, $|b| \leqslant r$.

Propriété P_5. $f(x)$ *est continue au point* $x = 0$.

Démonstration. Supposons qu'un r intervenant dans la propriété P_1 n'existe pas. Il existe donc trois suites de points $\{q_{1,j}\}$, $\{q_{2,j}\}$, $\{q_{3,j}\}$ telles que

$$q_{i,j} \to 0 \quad \text{lorsque} \quad j \to \infty \quad (i = 1, 2, 3)\,, \tag{4}$$

*) F est linéaire lorsqu'elle est continue pour $x=0$ et satisfait partout à l'équation $F(a+b) = F(a) + F(b)$.

$$(5)\qquad q_{1,j}+q_{2,j}+q_{3,j}=0 \qquad f(q_{1,j})+f(q_{2,j})+f(q_{3,j})\neq 0 \qquad (j=1,2,3)\,.$$

En choissant le nombre naturel n_j suffisamment grand on aura

$$(6)\qquad |n_j\{f(q_{1,j})+f(q_{2,j})+f(q_{3,j})\}|>1\,, \qquad (j=1,2,\ldots)\,.$$

Au moyen des q_{ij} nous construirons une série convergente $\sum_{k=1}^{\infty} b_k$ telle que $\sum_{k=1}^{\infty} f(b_k)$ sera divergente ce qui n'est pas compatible avec la propriété A. Posons $s_0=0$, $s_j=3n_1+3n_2+\ldots+3n_j$. Désignons par U_j la classe des indices k pour lesquels

$$1+s_{j-1}\leqslant k\leqslant s_j\,.$$

Nous repartirons U_j en trois classes disjointes $U_{1,j}$, $U_{2,j}$, $U_{3,j}$ en désignant par $U_{i,j}$ $(i=1,2,3)$ la classe de ces indices $k\in U_j$ qui peuvent être écrits sous la forme $k=3p+i$ où p est un nombre entier $(p\geqslant 0)$ et $1\leqslant i\leqslant 3$.

Chaque nombre naturel k appartient à une seule classe $U_{i,j}$. Nous définissons la suite $\{b_k\}$ en posant

$$(7)\qquad b_k=q_{i,j} \qquad \text{lorsque} \qquad k\in U_{i,j}\,.$$

De là il vient, en vertu de (4), que $b_k\to 0$ lorsque $k\to\infty$. Mais en vertu de (5) et (7) on a

$$\sum_{m=1}^{k} b_m=\sum_{m=k-i+1}^{k} b_m \qquad \text{lorsque} \qquad k\in U_{i,j}\,.$$

Comme la somme figurant dans le deuxième membre contient trois termes au plus, donc elle tend vers zéro lorsque $k\to\infty$. On a donc $\sum_{m=1}^{\infty} b_m=0$ et par suite la série $\sum_{m=1}^{\infty} f(b_m)$ est aussi convergente, ce qui est impossible, car on vérifie facilement (cf. (6) et la définition de b_k) que

$$\Big|\sum_{m=1+s_{j-1}}^{s_j} f(b_m)\Big|=|n_j[f(q_{1j})+f(q_{2j})+f(q_{3j})]|>1$$

et le premier terme de cette inégalité tend vers zéro lorsque $j\to\infty$ (puisque la série $\sum_{m=1}^{\infty} f(b_m)$ est convergente et $s_j\to\infty$ lorsque $j\to\infty$). La propriété P_1 se trouve ainsi établie.

En posant, dans l'énoncé de la propriété P_1, $q_1=q_2=q_3=0$ on obtient $3f(0)=0$ d'où résulte P_2. En posant $q_1=x$, $q_2=-x$, $q_3=0$ on obtient, en vertu de P_1 et de P_2, $f(x)+f(-x)=f(0)=0$ d'où vient P_3.

En posant $q_1 = a$, $q_2 = b$, $q_3 = -(a+b)$, nous obtiendrons pour $|a|$ et $|b|$ suffisamment petits $f(a)+f(b)+f(-(a+b)) = 0$, d'où, en vertu de P_3, résulte P_4.

Pour démontrer P_5 admettons que $x_i \to 0$. On a $|x_i| \leqslant r$ lorsque $i \geqslant N$ (N indice suffisamment grand). La suite

$$x_N, 0, x_{N+1}, 0, x_{N+2}, 0, \ldots \tag{8}$$

tend vers zero. Il en résulte que la série

$$x_N + (-x_N) + x_{N+1} + (-x_{N+1}) + x_{N+2} + (-x_{N+2}) + \ldots$$

dont les sommes partielles consécutives fournissent la suite (8), est convergente. Il en résulte que la série

$$f(x_N) + f(-x_N) + f(x_{N+1}) + f(-x_{N+1}) + \ldots$$

est convergente. Elle converge vers zéro, car ses sommes partielles paires sont, en vertu de P_3, égales à zéro. La suite des sommes partielles impaires $f(x_N), f(x_{N+1}), f(x_{N+2}), \ldots$ tend donc aussi vers zéro, c'est-à-dire (cf. P_2) vers $f(0)$. De la relation $x_i \to 0$ il vient donc que $f(x_i) \to f(0)$ ce qui prouve que P_5 a lieu.

Corollaire 1. *Il résulte du Théorème* 1 *que, si* $f(x)$ *possède une des propriétés* A, B, C, *alors à toute série convergente* (2) *correspond un indice* N, *tel que*

$$\sum_{j=1}^{\infty} f(x_j) = \sum_{j=1}^{N-1} f(x_j) + f\Big(\sum_{j=N}^{\infty} x_j\Big); \qquad \sum_{j=N}^{\infty} f(x_j) = f\Big(\sum_{j=N}^{\infty} x_j\Big).$$

Remarque 2. Dans l'Hypothèse H la condition nécessaire et suffisante pour que l'opération (1) transforme chaque série absolument convergente (2) en série (3) absolument convergente consiste en ce que l'on ait

$$f(0) = 0, \qquad \limsup_{x \to 0} \frac{|f(x)|}{|x|} < +\infty.$$

La démonstration est plus simple que celle du Lemme 1.

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Electron-Positron Pair Annihilation at Arbitrary Polarizations

by
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Presented by L. INFELD on June 24, 1958

Since the parity conservation law became invalidated in 1957, research work has been directed towards assembling the fullest possible experimental evidence proving or disproving invariance with regard to space inversion, as also with regard to charge conjugation and time reversal. In these investigations, polarization measurements play a considerable part.

In the present paper a method of measuring the polarization of positrons from β^+ decay will be discussed. The method uses the effect of annihilation of positrons with electrons, and was employed by S. S. Hanna and R. S. Preston [1], [2], who investigated the annihilation of a beam of positrons from a β^+ source on the electrons of a ferromagnetic target placed in a magnetic field. The cross-section for the annihilation of longitudinally or transversely polarized particles in the centre of a mass systems, was calculated by L. A. Page [3].

General formula for cross-section

An expression will now be derived yielding the cross-section in an arbitrary system of reference, for the case of two-quantum annihilation of arbitrarily polarized electron and positron beams, when the polarization of the photons is not observed. The formalism of scattering matrices will be used.

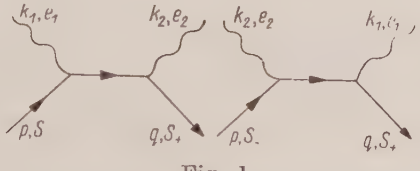

Fig. 1

The first approximation of the process considered is described by the graphs of Fig. 1. Assume the electron to be described in the initial

state by the four-momentum $p = (\vec{p}, E_-)$ and polarization vector S_-, the positron – by the four-momentum $q = (\vec{q}, E_+)$ and polarization vector S_+, and the photons in the final state by the momenta $k_1 = (\vec{k_1}, \omega_1)$ and $k_2 = (\vec{k}_2, \omega_2)$ and polarization vectors e_1 and e_2. With this notation, the scattering matrix element corresponding to the above graphs is:

$$\begin{aligned} &\langle k_1, e_1; k_2, e_2|S|p, S_-; q, S_+\rangle = \\ &= i\frac{me^2}{(2\pi)^2}\frac{1}{2\sqrt{\omega_1\omega_2 E_- E_+}}\bar{v}(q, S_+)Mu(p, S_-)\delta(p+q-k_1-k_2), \end{aligned} \tag{1}$$

wherein

$$M = \hat{e}_2\frac{i(\hat{p}-\hat{k}_1)-m}{(p-k_1)^2+m^2}\hat{e}_1 + \hat{e}_1\frac{i(\hat{p}-\hat{k}_2)-m}{(p-k_2)^2+m^2}\hat{e}_2 \tag{2}$$

and u, v are spinors fulfilling Dirac's equation:

$$(i\hat{p}+m)u = 0 \qquad (i\hat{q}-m)v = 0 \tag{3}$$

with $\bar{v} = v^+A$, the matrix A being chosen so that $\gamma_\mu^+ = -A\gamma_\mu A^{-1}$. The Dirac matrices satisfy the relation: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, where $g_{00} = -1$. The symbol $\hat{p}$ stands for $\hat{p} = p^\mu\gamma_\mu$. The polarization vectors are defined as follows:

$$S_-^\mu = -\bar{u}(p, S_-)\gamma_5\gamma^\mu u(p, S_-), \qquad S_+^\mu = -\bar{u}_c(q, S_+)\gamma_5\gamma^\mu u_c(q, S_+), \tag{4}$$

where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, and u_c is a charge conjugated spinor.

If the polarization of the photons is not observed, by (1), the differential cross-section is:

$$\begin{aligned} d\sigma(p, S_-; q, S_+; k_1; k_2) = r_0^2\frac{m^4}{\omega_1\omega_2\sqrt{(pq)^2-m^4}}\delta(p+q-k_1-k_2)\times \\ \times\sum_{e_1, e_2}|\bar{v}(q, S_+)Mu(p, S_-)|^2 d_3k_1 d_3k_2, \end{aligned} \tag{5}$$

where $r_0 = \dfrac{e^2}{4\pi m}$ is the classical radius of the electron.

Summation over the polarizations of the photons yields

$$\sum_{e_1, e_2}|\bar{v}(q, S_+)Mu(p, S_-)|^2 = \bar{v}(q, S_+)L^{\mu\nu}u(p, S_-)\bar{u}(p, S_-)L_{\nu\mu}v(q, S_+), \tag{6}$$

where

$$L^{\mu\nu} = \gamma^\mu\frac{i(\hat{p}-\hat{k}_1)-m}{(p-k_1)^2+m^2}\gamma^\nu + \gamma^\nu\frac{i(\hat{p}-k_2)-m}{(p-k_2)^2+m^2}\gamma^\mu.$$

On inserting the corresponding projection operators in (6), expression (5) takes the form

$$\begin{aligned} d\sigma = -r_0^2\frac{m^4}{\omega_1\omega_2\sqrt{(pq)^2-m^4}}\delta(p+q-k_1-k_2)\times \\ \times\operatorname{Tr}\{L^{\mu\nu}\lambda(S_-)\Lambda_-(p)L_{\nu\mu}\lambda(S_+)\Lambda_+(q)\}d_3k_1 d_3k_2, \end{aligned} \tag{7}$$

where

$$\Lambda_{\pm}(p) = \frac{\pm i\hat{p} + m}{2m}, \quad \lambda(S_{\pm}) = \tfrac{1}{2}(1 - \gamma_5 \hat{S}_{\pm}).$$

The calculation gives

$$\begin{aligned}
&\mathrm{Tr}\{L^{\mu\nu}\lambda(S_-)\Lambda_-(p)L_{\nu\mu}\lambda(S_+)\Lambda_+(q)\} = \\
&= \frac{1}{2m^2}\left\{\left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1}\right)\left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1} + 2\right) - \left(\frac{pk_1}{qk_1} + \frac{qk_1}{pk_1}\right) + \right. \\
&+ (S_-S_+)\left[\left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1}\right)\left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1} + 2\right) + 2\right] + \\
&\left. + \left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1} + 2\right)\left[(k_1S_-)(k_1S_+)\left(\frac{1}{pk_1} + \frac{1}{qk_1}\right) - \frac{(k_1S_+)(qS_-)}{qk_1} - \frac{(k_1S_-)(pS_+)}{pk_1}\right]\right\}.
\end{aligned} \tag{8}$$

After getting rid of the conservation δ-function in expression (7) one obtains the differential cross-section for the annihilation of an electron and a positron of given momenta and polarizations with emission of one of the photons in the direction of the elementary solid angle $d\Omega_1$:

$$d\sigma(p, S_-; q, S_+) = \frac{r_0^2 m^2}{\sqrt{(pq)^2 - m^4}} \frac{\omega_1}{\omega_2} \left|\frac{d\omega_1}{d(\omega_1 + \omega_2)}\right| X(p, S_-; q, S_+) d\Omega_1, \tag{9}$$

where

$$X(p, S_-; q, S_+) = -m^2 \mathrm{Tr}\{L^{\mu\nu}\lambda(S_-)\Lambda_-(p)L_{\nu\mu}\lambda(S_+)\Lambda_+(q)\}|_{p+q=k_1+k_2}, \tag{10}$$

the derivative $\frac{d\omega_1}{d(\omega_1+\omega_2)}$ being computed from the relation $\vec{p} + \vec{q} = \vec{k}_1 + \vec{k}_2$ and taken at the point for which $\omega_1 + \omega_2 = E_+ + E_-$. If the polarization of the particles in the initial state is not observed, summation and averaging over the polarizations S_- and S_+ in expression (9) should be carried out. In this case, by (8) and (10), the following expression for the differential cross-section is obtained:

$$d\sigma_0(p, q) = \frac{r_0^2 m^2 d\Omega_1}{2\sqrt{(pq)^2 - m^4}} \frac{\omega_1}{\omega_2} \left|\frac{d\omega_1}{d(\omega_1 + \omega_2)}\right| \left\{\frac{pk_1}{qk_1} + \frac{qk_1}{pk_1} - \left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1}\right) \times \left(\frac{m^2}{pk_1} + \frac{m^2}{qk_1} + 2\right)\right\} \tag{11}$$

in agreement with J. M. Jauch and F. Rohrlich [4].

Expression for cross-section in the laboratory system

Expressions (9) and (11) will now be presented in a co-ordinate system which, in measurements of the polarization of the positron, is usually the laboratory system, and will be specialized for certain cases of polarization.

Assume a system in which the electron is at rest, i. e.:

$$p = (\vec{0}, m) \tag{12}$$

and the positron moves with velocity β in the direction of the vector $\vec{j}$, i. e.

$$q = (m\beta\gamma\vec{j}, m\gamma), \tag{13}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\vec{j}^2 = 1$.

In this system, the polarization vector of the electron is of the form

$$S_- = (\vec{s}_-, 0), \tag{14}$$

where $\vec{s}_- = u^+(\vec{0}, \vec{s}_-)\vec{\sigma}u(\vec{0}, \vec{s}_-)$ is the spin vector. The vector of the polarization of the positron is related to that of the spin $\vec{s}_+ = u_c^+(\vec{0}, \vec{s}_+)\vec{\sigma}u_c(\vec{0}, \vec{s}_+)$ as follows:

$$S_+ = \left(\vec{t}_+ + \gamma\vec{l}_+, \frac{\vec{q}\,\vec{l}_+}{m}\right), \tag{15}$$

where $\vec{s}_+ = \vec{l}_+ + \vec{t}_+$ with $\vec{t}_+ \perp \vec{j}$, $\vec{l}_+ \| \vec{j}$.

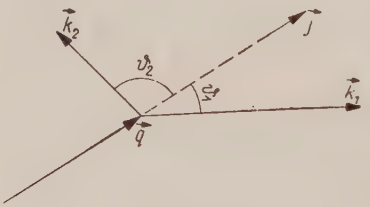

Fig. 2

Assume the photon of momentum $\vec{k}_1$ to be emitted at the angle ϑ_1 to the direction of motion of the positron (Fig. 2), i. e. $\vec{k}_1\vec{j} = \omega_1\cos\vartheta_1$. Then, conservation of energy and momentum leads to

$$\frac{d(\omega_1+\omega_2)}{d\omega_1} = \frac{(1+\gamma(1-\beta\cos\vartheta_1))^2}{(1+\gamma)\gamma(1-\beta\cos\vartheta_1)}. \tag{16}$$

If the polarizations of the particles are not observed, the cross-section in the co-ordinate system considered, by (12), (13) and (16), takes the form *):

$$d\sigma_0 = \frac{r_0^2}{2\beta\gamma}\frac{d\Omega_1}{\gamma(1-\beta\cos\vartheta_1)}\left\{(1+\gamma)+2-\frac{2(1+\gamma)\gamma(1-\beta\cos\vartheta_1)}{(1+\gamma(1-\beta\cos\vartheta_1))^2} - \right. \tag{17}$$
$$\left. - \frac{(1+\gamma(1-\beta\cos\vartheta_1))^2}{(1+\gamma)\gamma(1-\beta\cos\vartheta_1)}\right\}.$$

*) The expression derived above does not coincide with that given by Jauch and Rohrlich, which contains two inaccuracies in the signs and hence yields a negative cross-section for $\beta \to 0$.

1. Transversal polarization. The case will now be considered of the spin vector of the electron perpendicular to the plane of reaction (Fig. 3), i. e. $\vec{s}_- = \vec{t}_-$, where $\vec{j}\vec{t}_- = \vec{k}_1\vec{t}_- = 0$ (the direction of the spin of the electron coincides with that of the magnetic field in the experi-

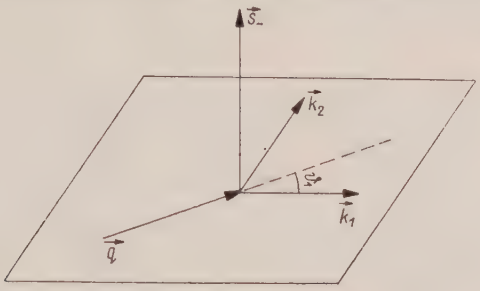

Fig. 3

ments of Hanna and Preston). The vector of the spin of the positron $\vec{s}_+ = \vec{l}_+ + \vec{t}_+$ may be directed arbitrarily.

Now, by (12), (13), (14), (15) and (16), the cross-section (9) takes the form

$$d\sigma = d\sigma_0 + (\vec{t}_+\vec{t}_-)\,d\sigma_t, \tag{18}$$

where

$$d\sigma_t = \frac{r_0^2}{2\beta\gamma}\,\frac{d\Omega_1}{\gamma(1-\beta\cos\vartheta_1)}\left\{2 - \frac{2(1+\gamma)\gamma(1-\beta\cos\vartheta_1)}{(1+\gamma(1-\beta\cos\vartheta_1))^2} - \right. \tag{19}$$
$$\left. - \frac{(1+\gamma(1-\beta\cos\vartheta_1))^2}{(1+\gamma)\gamma(1-\beta\cos\vartheta_1)}\right\}.$$

The effect of the polarization on the annihilation cross-section is described by the quantity $\frac{d\sigma_t}{d\sigma_0}$. A large value of $\left|\frac{d\sigma_t}{d\sigma_0}\right|$ means the strong dependence of the cross-section on the direction of the transversal components of the spin vectors. The sign of $\frac{d\sigma_t}{d\sigma_0}$ determines whether parallel or anti-parallel polarizations are privileged in the process.

It will be seen from the graphs in Fig. 4 that, for arbitrary β and angles ϑ_1, the quantity $\frac{d\sigma_t}{d\sigma_0} < 0$, and hence the cross-section takes its largest value if the transversal spin components of both particles have opposite directions, and is smallest if the latter have the same orientation. For low energies of the positrons (not exceeding 1 MeV), $\frac{d\sigma_t}{d\sigma_0}$ is considerable (larger than 0.5) and weakly dependent on ϑ_1. Hence, the cross-section is sensitive to the orientation of the transversal spin components at all values of ϑ_1. For higher energies, the effect of the polari-

zation is, in general, smaller. However, from 1 MeV upwards, the quantity $\left|\frac{d\sigma_t}{d\sigma_0}\right|$ exhibits a maximum of growing steepness for $\vartheta_1 = \arccos\frac{\gamma\beta}{\gamma+1}$, i. e. for $\omega_1 = \omega_2$. Thus, for the same values of the energies of the emitted quanta, the cross-section is still strongly dependent on the polarization. Thus, we see that the expression (19) may be used for measuring the

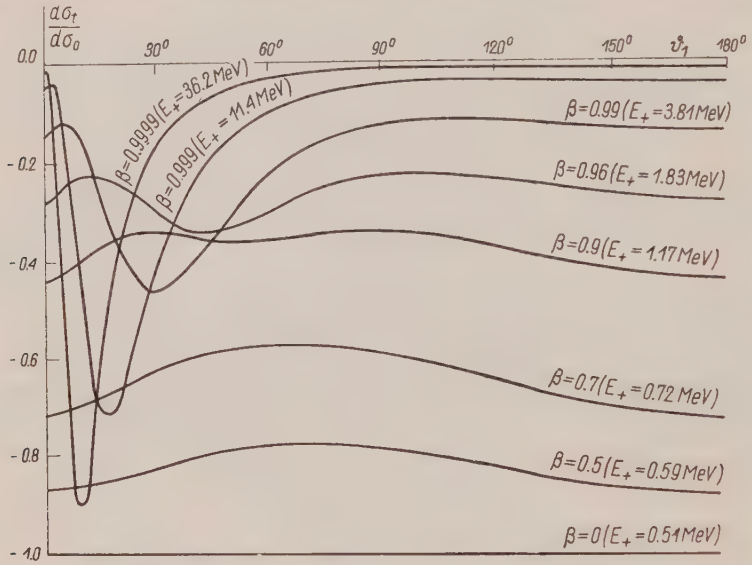

Fig. 4. The quantity $\frac{d\sigma_t}{d\sigma_0}$ for transversal polarization

transversal components of the spin vector of a positron of arbitrary energy. The conditions for such a measurement are relatively most disadvantageous at energies of 1-2 MeV (as then $\left|\frac{d\sigma_t}{d\sigma_0}\right| < 0.5$ for all angles ϑ_1).

2. Longitudinal polarization. Now assume the spin vector of the electron to be parallel to the momentum of the positron: $\vec{s}_- = \vec{l}_-$, where $\vec{l}_- \| \vec{j}$. If the positrons are polarized longitudinally, $\vec{t}_+ = 0$, then, by (12), (13), (14), (15) and (16), the expression (9) takes the form

$$d\sigma = d\sigma_0 + (\vec{l}_-\vec{l}_+)\,d\sigma_t, \tag{20}$$

where

$$d\sigma_t = \frac{r_0^2}{2\beta\gamma}\,\frac{d\Omega_1}{\gamma(1-\beta\cos\vartheta_1)}\Big\{\gamma+\gamma(1-\beta\cos\vartheta_1)- \\ -2(1+\gamma)\,\frac{1+\gamma\beta\cos\vartheta_1-(1+\gamma)\cos^2\vartheta_1}{(1+\gamma(1-\beta\cos\vartheta_1))^2} - \frac{(1+\gamma(1-\beta\cos\vartheta_1))^2}{(1+\gamma)(1-\beta\cos\vartheta_1)} - \\ -\frac{\gamma\beta^2-(1+2\gamma)\beta\cos\vartheta_1+(1+\gamma)\cos^2\vartheta_1}{1-\beta\cos\vartheta_1}\Big\}. \tag{21}$$

Here, the effect of the polarization on the cross-section is determined by the quantity $\frac{d\sigma_\iota}{d\sigma_0}$. Fig. 5 shows that for low energies of the positrons

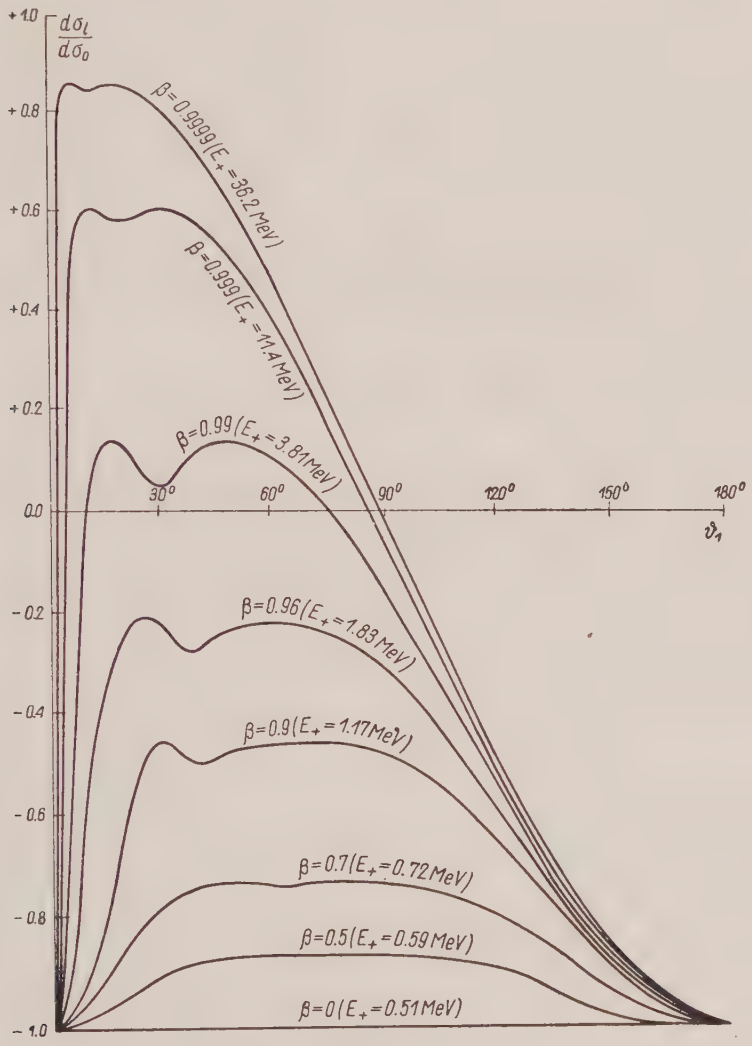

Fig. 5. The quantity $\frac{d\sigma_\iota}{d\sigma_\parallel}$ for longitudinal polarization

(up to 1.2 MeV) the cross-section depends strongly on the orientation of the spins for all angles ϑ_1, anti-parallel orientation being privileged. For positrons of several MeV, the cross-section is strongly dependent on the polarization only for very small angles (near 0°) and for angles of 120° to 180° (as then $\left|\frac{d\sigma_\iota}{d\sigma_0}\right| > 0.5$). For high energies (upwards of 10 MeV), the process is polarization sensitive also for angles of 10°-60°; however, it is the parallel spin orientation that is privileged in this case.

For angles $\vartheta_1 = 0^0$ and $\vartheta_1 = 180^0$ we have $\frac{d\sigma_t}{d\sigma_0} = -1$ at all energies of the positrons. Hence, for parallel spin orientation and total polarization of the particles no quanta are emitted in the forward or backward direction, as then $d\sigma = 0$.

If transversal polarization of the positrons is taken into account ($\vec{t}_+ \neq 0$), two components of the spin vector: $\vec{l}_+$ and the projection of $\vec{t}_+$ upon the plane of the reaction appear in (20). To determine both components simultaneously, it is necessary to measure the cross section for two angles ϑ_1.

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On Gravitational Radiation Damping

by

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Presented by L. INFELD on July 28, 1958

1. The problem of gravitational radiation has several aspects. In this paper those which are related to the free motion of gravitating bodies are considered.

In Newtonian mechanics the initial positions and velocities of point-masses determine their motion completely. The situation is different in electrodynamics where the initial values of the field are required besides information concerning changes. Two *free* point-charges of opposite signs may move uniformly around a circle in a standing-wave electromagnetic field. However, the same charges may alternatively produce outgoing radiation. Their motion will not then be periodic; they will undergo damping. Which of these cases occurs in any particular system depends on the initial and boundary conditions.

The question arises as to whether the situation in General Relativity resembles that in Newtonian mechanics rather than that in electrodynamics. As yet, the most powerful tool for dealing with the problem of motion is the Einstein-Infeld-Hoffmann "new" approximation method [1]. This provides a description of a set of motions which are compatible with the field equations and which, in the first significant order, correspond to the Newtonian motions. The original EIH method is based on certain assumptions corresponding to the choice of symmetric potentials in electrodynamics. Uniform circular motion of two bodies of equal mass is possible within the original framework of the new approximation method, but "damped" motion along a spiral path is not. The problem of whether the EIH method discloses *all* possible motions of bodies interacting through the gravitational field has been the object of several papers. Infeld [2] supplemented the original EIH expansions by certain "radiation terms" and showed that they do not contribute to the equations of motion of the 7th order (the Newtonian equations being of the 4th). Hu [3] performed the next step of the approximation

procedure and found the radiative corrections to the equations of motion in the 9th order. The result of his lengthy calculations was paradoxical: it turned out that the total energy of a radiating double-star system would increase. Infeld and Scheidegger [4] investigated the possibility of eliminating the radiation terms by means of co-ordinate transformations. Later, Goldberg [5] found radiative solutions which cannot be annihilated in this way.

The aim of the present paper is to show that there are free gravitational motions which differ from those described by the original formulation of the EIH method. As an illustration, the calculations referring to the problem of two bodies of equal mass are performed by a method slightly different from that of Hu. On the assumption that the Newtonian motion is circular, the equations of motion up to the 9th order turn out to be of the type $\ddot{x} + 2a\dot{x} + \omega^2(x)x = 0$, where $a = \text{const} > 0$.

2. The EIH method is based on a special approximation procedure and provides a way of obtaining the equations of motion. The fact that the equations of motion follow from the field equations is due to the general covariance of Einstein's theory, but the new approximation method can also be applied in other field theories.

Let $\varphi = \varphi(x^\nu, c)$ denote a function of co-ordinates x^ν $(\nu = 0, 1, 2, 3)$ depending on a parameter c. In the new approximation method a new time variable $t = x^0/c$ is introduced and it is assumed that φ can be expanded in a power series in $1/c$.

$$\varphi = \varphi(ct, x^k, c) = \sum_{m=0}^{\infty} c^{-m} \underset{m}{\varphi}(t, x^k), \qquad (k = 1, 2, 3). \tag{1}$$

It follows that $\partial\varphi/\partial x^0 = c^{-1}\partial\varphi/\partial t = c^{-1}\dot{\underset{m}{\varphi}}$ is of order $m+1$. We choose $c = 1$, bearing in mind subsequently that differentiation with respect to time increases the order of a term by one.

If φ satisfies the wave equation $\Box\varphi = 0$, then the fields $\underset{m}{\varphi}$ are subject to the relations

$$\Delta\underset{0}{\varphi} = 0, \quad \Delta\underset{1}{\varphi} = 0, \quad \Delta\underset{2}{\varphi} = \ddot{\underset{0}{\varphi}}, \ \ldots, \quad \Delta\underset{m}{\varphi} = \ddot{\underset{m-2}{\varphi}}, \ \ldots, \tag{2}$$

The retarded spherical wave

$$\varphi = a(t-r)/r = \sum_{m=0}^{\infty} (-1)^m (m!)^{-1} r^{1-m} d^m a(t)/dt^m \tag{3}$$

is a solution of (2). By omitting the odd terms in (3) one obtains the half-advanced, half-retarded wave $\frac{1}{2}[a(t-r) + a(t+r)]$. In order to make easier the comparison with the original works on the EIH method $a(t)$ is assumed to be of second order. The odd terms in (3) are called "radiation

terms", the first of which, $\underset{3}{\varphi} = -\dot{a}(t)$, is a function of time alone. It is important to note that $\underset{m}{\varphi} = 0(r^{m-3})$ for large r, this being a general property of solutions of the inhomogeneous wave equation with a spatially bounded source. If the field φ is periodic and if λ denotes the corresponding wave-length, then the ratio of two consecutive terms in (1) is of order r/λ (where r is the distance from the source). The first few terms of the series (1) will closely approximate the field φ only in the region $r \ll \lambda$; the EIH method is not well-suited to the investigation of the field in the wave zone.

In *electrodynamics* the situation is quite similar, and we shall only stress certain peculiarities due to gauge-invariance. If A^α is a potential satisfying the Lorentz condition $A^\alpha{}_{,\alpha} = 0$, then Maxwell's equations read

$$\Box A^\alpha = -4\pi j^\alpha, \qquad j^\alpha{}_{,\alpha} = 0\,. \tag{4}$$

It is convenient to assume that j^0 and j^k are of the second and third orders, respectively. For regular and spatially bounded j^α we may solve (4) using, e. g., the retarded Green function. On expanding the solution into an EIH power series we find that the first radiation term

$$\underset{3}{A^0} = -\int j^0{}_{,0}\, dV$$

vanishes because of the conservation of charge: $j^0{}_{,0} + j^k{}_{,k} = 0$. Thus the first non-vanishing radiation term is of the 4th order, $\underset{4}{A^k}$. For large , and for $m \geqslant 3$ we have $\underset{m}{A^\alpha} = 0(r^{m-4})$.

An analogous discussion is possible in the *linearized theory of gravitation*. If we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (with $h_{\mu\nu}$ small), $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}h_{\varrho\sigma}$ and assume $\gamma^{\mu\nu}{}_{,\nu} = 0$, then Einstein's equations become

$$\Box\gamma^{\mu\nu} = 16\pi T^{\mu\nu}, \qquad T^{\mu\nu}{}_{,\nu} = 0\,, \tag{5}$$

where terms non-linear in $h_{\alpha\beta}$ have been neglected. The functions T^{00}, T^{0k} and T^{ik} can be assumed to be of second, third and fourth orders, respectively. The $\underset{3}{\gamma_{00}}$ and $\underset{4}{\gamma_{0k}}$ vanish because of the conservation laws [2], hence $\underset{5}{\gamma_{00}}$, $\underset{5}{\gamma_{ik}}$ are the first non-vanishing radiation terms.

Taking into account the conservation laws one may easily show that, for large r and for $m \geqslant 4$, $\underset{m}{\gamma_{\mu\nu}}$ behaves like r^{m-5}.

3. The structure of *Einstein's equations* is such that we can, if we wish, choose solutions of the form

$$\begin{aligned} g_{00} &= 1 + \underset{2}{g_{00}} + \underset{4}{g_{00}} + \underset{6}{g_{00}} + \cdots \\ g_{0k} &= \underset{3}{g_{0k}} + \underset{5}{g_{0k}} + \underset{7}{g_{0k}} + \cdots \\ g_{ik} &= -\delta_{ik} + \underset{2}{g_{ik}} + \underset{4}{g_{ik}} + \underset{6}{g_{ik}} + \cdots\,. \end{aligned} \tag{6}$$

By analogy with the scalar wave equation and Maxwell's theory, solutions of the form (6) may be interpreted as representing standing-wave fields. In order to get solutions corresponding to "retarded" or "advanced" fields the series (6) must be supplemented with the missing *radiation terms*, i. e. those odd in g_{00} and g_{ik} and even in g_{0k}.

A solution for $\underset{n}{g_{\mu\nu}}$ $(n \geqslant 4)$ contains, in general, both terms linear and non-linear in the masses. The linear part of $\underset{n}{g_{\mu\nu}}$ is easily calculated from the linearized theory by solving (5). We may expect $\underset{n}{g_{\mu\nu}}$ also to behave like r^{n-5} $(n \geqslant 4)$, unless some of the non-linear terms in $\underset{n}{g_{\mu\nu}}$ cancel with the r^{n-5} terms in the linear part. In general, one cannot impose on the expanded metric the condition $\lim\limits_{r\to\infty} \underset{n}{g_{\mu\nu}} = 0$. However, this does not necessarily mean that the metric is non-flat at infinity.

The first radiation terms satisfy linear homogeneous equations and we may expect them to be linear in the masses so that their form can be derived from the linearized theory. If we take $T^{\mu\nu}$ in the form proposed by Infeld [6] for spherically symmetric non-rotating bodies

$$\sqrt{-g}T^{\mu\nu} = \sum m\dot{\xi}^{\mu}\dot{\xi}^{\nu}\delta_{(3)}\big(x^s - \xi^s(t)\big)\,dt/ds, \qquad m = \underset{2}{m} = \text{const.}, \tag{7}$$

choose a retarded solution of (5), and expand it into an EIH series, then the first non-vanishing radiation terms become

$$\underset{5}{\gamma_{00}} = \frac{4}{3!}\sum \frac{d^3}{dt^3}(mr^2), \qquad \underset{5}{\gamma_{ik}} = 4\sum \frac{d}{dt}(m\dot{\xi}^i\xi^k),$$

$$\underset{6}{\gamma_{0k}} = -4\sum m\ddot{\xi}^k - \frac{4}{3!}\sum \frac{d^3}{dt^3}(mr^2\dot{\xi}^k), \tag{8}$$

where summation is to be taken over all particles. Here, the functions $\xi^k(t)$ are the co-ordinates of a typical particle, $\xi^0 = t$, and r is the Euclidean distance from the particle. The first sum in $\underset{6}{\gamma_{0k}}$ is of at least the 6th order because of the Newtonian equations of motion [2]. The $\underset{5}{g_{00}}$, $\underset{5}{g_{ik}}$ and $\underset{6}{g_{0k}}$ corresponding to (8) can be obtained from the formula

$$\underset{n}{g_{\mu\nu}} = \underset{n}{\gamma_{\mu\nu}} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}\underset{n}{\gamma_{\varrho\sigma}}.$$

If a *co-ordinate transformation*

$$x^0 = x'^0 + a^0(x'^{\nu}), \qquad x^k = x'^k + a(x'^{\nu}),$$

is performed then the first terms affected are

$$\underset{n+2}{g'_{00}} = \underset{n+2}{g_{00}} + \underset{n+1}{2a_{0,0}}, \qquad \underset{n+1}{g'_{0k}} = \underset{n+1}{g_{0k}} + \underset{n+1}{a_{0,k}} + \underset{-n}{a_{k,0}},$$

$$\underset{n}{g'_{ik}} = \underset{n}{g_{ik}} + \underset{n}{a_{i,k}} + \underset{n}{a_{k,i}}, \qquad a_{\alpha} = \eta_{\alpha\beta}a^{\beta}.$$

It can easily be seen that, if $(\underset{n}{g_{ik}}, \underset{n+1}{g_{0k}}, \underset{n+2}{g_{00}})$ is a solution of the field equations, then $(\underset{n}{g'_{ik}}, \underset{n+1}{g'_{0k}}, \underset{n+2}{g'_{00}})$ is also a solution representing the same physical situation in a different co-ordinate system. Because of the Newtonian equations, the functions $\underset{5}{g_{ik}}$ depend on the time only, and thus one can choose $\underset{5}{a_k}$ such that $\underset{5}{g'_{ik}} = 0$. However, the whole set of functions $(\underset{n}{g_{ik}}, \underset{n+1}{g_{0k}}, \underset{n+2}{g_{00}})$ can be annihilated by a co-ordinate transformation only if

$$\underset{n+2}{g_{00,ik}} + \underset{n}{g_{ik,00}} - \underset{n+1}{g_{i0,k0}} - \underset{n+1}{g_{k0,i0}} = 0\,, \tag{9a}$$

$$\underset{n+1}{g_{0m,ik}} + \underset{n}{g_{ik,0m}} - \underset{n+1}{g_{0i,km}} - \underset{n}{g_{km,0i}} = 0\,, \tag{9b}$$

$$\underset{n}{g_{im,kl}} + \underset{n}{g_{kl,im}} - \underset{n}{g_{il,km}} - \underset{n}{g_{km,il}} = 0\,. \tag{9c}$$

That is to say, Eqs. (9) constitute a system of necessary and sufficient conditions for the existence of functions $\underset{n+1}{a_0}$ and $\underset{n}{a_k}$ such that $\underset{n}{g'_{ik}} = \underset{n+1}{g'_{0k}} = \underset{n+2}{g'_{00}} = 0$. Goldberg [5] remarked that by starting with $\underset{n}{g_{ik}} = f_{ik}(t)$ one can obtain solutions of the field equations of the $(n+1)$th and $(n+2)$th orders such that conditions (9) will not be satisfied. However, it must be noted that, since the solutions of the field equations are not unique, one can also start with the same $\underset{n}{g_{ik}}$ and obtain functions $\underset{n+1}{g_{0k}}$ and $\underset{n+2}{g_{00}}$ which can be annihilated. For example, the field

$$\underset{n}{g_{ik}} = f_{ik}(t), \qquad \underset{n+1}{g_{0k}} = \tfrac{1}{2} x^s \dot{f}_{sk}\,, \qquad \underset{n+2}{g_{00}} = 0$$

is flat, i. e. it fulfills conditions (9), but the field

$$\underset{n}{g_{ik}} = f_{ik}(t), \qquad \underset{n+1}{g_{0k}} = 0, \qquad \underset{n+2}{g_{00}} = -r^2 \ddot{f}_{ss}/6$$

is empty and non-flat unless $\ddot{f}_{ik} = \frac{1}{3}\delta_{ik}\ddot{f}_{ss}$ (spherical symmetry), that is

$$\underset{n+2}{g_{00,ik}} + \underset{n}{g_{ik,00}} - \underset{n+1}{g_{i0,k0}} - \underset{n+1}{g_{k0,i0}} = \ddot{f}_{ik} - \tfrac{1}{3}\delta_{ik}\ddot{f}_{ss}\,.$$

The radiation terms $(\underset{3}{g_{ik}} = 0, \underset{4}{g_{0k}} = 0, \underset{5}{g_{00}})$ represent an apparent field, i. e. they can be eliminated by a co-ordinate transformation and do not contribute to the equations of motion up to the 7th order [2]. This suggests that $(\underset{5}{g_{ik}}, \underset{6}{g_{0k}}, \underset{7}{g_{00}})$, should be the first true radiative set contributing to the equations of motion of the 9th order. In the case of a system of pole-particles, $\underset{5}{g_{ik}}$ and $\underset{6}{g_{0k}}$ can be determined from (8), and $\underset{7}{g_{00}}$ from the corresponding field equation

$$\underset{7}{R_{00}} = -\tfrac{1}{2}\Delta \underset{7}{g_{00}} + \tfrac{1}{2}\underset{5}{g_{00,00}} - \tfrac{1}{2}\underset{5}{g_{ik}}\underset{2}{g_{00,ik}} = -8\pi(\underset{7}{T_{00}} - \tfrac{1}{2}\underline{\underset{7}{g_{00}}\underset{7}{T}})\,, \tag{10}$$

where $\underset{2}{g}_{00}$ is given by the usual EIH expression $\underset{2}{g}_{00} = -\sum 2m/r$. The functions $\underset{7}{T}_{00}$ and $\underset{7}{T}$ can be evaluated from (7). One must note that $m\,dt/ds$ contains a 7th order term $\underset{7}{m} = -\frac{1}{2}m\underset{5}{g}_{00}$. Finally, as the solution of (10), we obtain the expression

(11)
$$\underset{7}{g}_{00} = \frac{2}{5!}\sum \frac{d^5}{dt^5}(mr^4) + \frac{2}{3!}\sum \frac{d^3}{dt^3}(mr^2\dot{\xi}^s\dot{\xi}^s) + \underset{5}{g}_{ik}\sum mr_{,ik} + (\underset{5}{g}_{00'} + \tfrac{1}{2}\underset{5}{g}_{ss})\underset{2}{g}_{00}\,.$$

The set of fields $(\underset{5}{g}_{ik}, \underset{6}{g}_{0k}, \underset{7}{g}_{00})$ defined by (8) and (11) cannot, in general, be eliminated by a co-ordinate transformation. This may be confirmed by calculating the expression (9a) for $n = 5$.

4. For spherically symmetric non-rotating particles the equations of motion can be obtained from the "geodesic" equation [7], [8]

(12)
$$m\left(\frac{d^2\xi^\alpha}{d\tilde{s}^2} + \widetilde{\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}}\frac{d\xi^\mu}{d\tilde{s}}\frac{d\xi^\nu}{d\tilde{s}}\right) = 0\,,$$

where $d\tilde{s}^2 = \tilde{g}_{\alpha\beta}d\xi^\alpha d\xi^\beta$, and where $\tilde{\varphi}$ denotes the regular part of φ with x^k replaced by ξ^k (see [8] for details). By eliminating $d\tilde{s}$ from (12) one obtains the equations of motion in the form

(13)
$$\Omega^k \equiv m\left(\ddot{\xi}^k + \left(\widetilde{\left\{\begin{matrix}k\\ \mu\nu\end{matrix}\right\}} - \widetilde{\left\{\begin{matrix}0\\ \mu\nu\end{matrix}\right\}}\dot{\xi}^k\right)\dot{\xi}^\mu\dot{\xi}^\nu\right) = 0\,.$$

In this notation the Newtonian equations are

$$\underset{4}{\Omega}^k \equiv m\left(\ddot{\xi}^k + \widetilde{\underset{2}{\left\{\begin{matrix}k\\ 00\end{matrix}\right\}}}\right) = 0\,.$$

The equations of motion can be written to the 9th order —

(14)
$$\underset{4}{\Omega}^k + \underset{6}{\Omega}^k + \underset{8}{\Omega}^k + \underset{9}{\Omega}^k = 0\,.$$

Einstein, Infeld and Hoffman [1] have found the first post-Newtonian contribution $\underset{6}{\Omega}^k$. The radiative correction has the form

(15)
$$\underset{9}{\Omega}^k \equiv m\left(\widetilde{\underset{7}{\left\{\begin{matrix}k\\ 00\end{matrix}\right\}}} + 2\widetilde{\underset{6}{\left\{\begin{matrix}k\\ 0s\end{matrix}\right\}}}\dot{\xi}^s + \widetilde{\underset{5}{\left\{\begin{matrix}k\\ rs\end{matrix}\right\}}}\dot{\xi}^r\dot{\xi}^s - \widetilde{\underset{6}{\left\{\begin{matrix}0\\ 00\end{matrix}\right\}}}\dot{\xi}^k - 2\widetilde{\underset{5}{\left\{\begin{matrix}0\\ 0s\end{matrix}\right\}}}\dot{\xi}^s\dot{\xi}^k - \widetilde{\underset{4}{\left\{\begin{matrix}0\\ rs\end{matrix}\right\}}}\dot{\xi}^r\dot{\xi}^s\dot{\xi}^k\right)$$

and can be explicitly evaluated by means of (8) and (11). The explicit form of $\underset{8}{\Omega}^k$ is not known, but it is possible to foresee it in special simple cases.

Let us consider the case of *two bodies* of equal mass m, with coordinates ξ^k and η^k, respectively. The equations of motion up to the 6th order admit a solution which represents the circular motion of these bodies. One may take $\xi^k = -\eta^k$ and

$$\xi^1 = R\cos\omega_0 t, \quad \xi^2 = R\sin\omega_0 t, \quad \xi^3 = 0\,. \tag{16}$$

The angular velocity ω_0 is a function of R and m which, in the Newtonian approximation, is

$$\omega_0^2 = m/4R^3\,. \tag{17}$$

From symmetry considerations it follows that the equations of motion to the 9th order have also a solution with $\xi^k = -\eta^k$. The 8th order terms are either of the form $A\xi^k$ or $B\dot{\xi}^k$; but $B = 0$ by virtue of (16). Thus, the equations to the 8th order also admit periodic solutions representing circular motion. It follows that the equations of motion up to the 9th order have the form

$$m(\ddot{\xi}^k + 2a\dot{\xi}^k + \omega^2\xi^k) = 0\,, \tag{18}$$

where a is a constant of the 6th order, and

$$\omega^2 = m/4|\vec{\xi}|^3 + \text{constants of order 4 and 6.} \tag{19}$$

The damping term $2am\dot{\xi}^k$ arises from Ω^k. The value of a is determined by the use of Eq. (15) with g_{ik}, g_{0k}, g_{00} given by (8) and (11). Thus,

$$a = 3m^3/20R^4. \tag{20}$$

Eq. (18) does not admit exact solutions of the form (16). In spite of the non-linearity of Eq. (18) one might expect it to admit solutions of a "damped" character ($a > 0$), at least for a limited time range.

I am grateful to Professor L. Infeld and Dr. J. Plebański for many interesting discussions during this investigation. The work was concluded at King's College, London, and I am indebted to Professor H. Bondi and Dr. F. A. E. Pirani for their hospitality there.

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THEORETICAL PHYSICS

Binding Energy of the ^{16}O Nucleus with the Cluster Development Method

by
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Presented by W. RUBINOWICZ on July 29, 1958

The ^{16}O nucleus is a convenient example for an investigation of nuclear binding and size. We start from realistic although simplified two body forces, namely we assume a central Yukawa interaction with a hard core of radius a and with Serber exchange. For each of the values of $a = 0.6\,f$ and $0.2\,f$, the parameters of the interaction were adjusted to fit the low energy data.

Although the formal scheme of the Bruckner theory is applicable to the finite nucleus (see e. g. [3]), the performance of the corresponding calculations would be extremely difficult. An attempt to overcome these difficulties has recently been made by Brueckner, Gammel and Weitzner [4] (see also [5]) who make the simplifying assumption that the local properties of a finite nucleus can be identified with those of infinite nuclear matter. Another attempt to solve the problem of the finite nucleus in the frame of the Brueckner theory has recently been reported by Eden and Emery [6] whose approximations concern, among others, the Pauli principle and the self-consistency. But any simplifying assumptions introduced into the scheme of the Brueckner theory are inevitably connected with the difficulty of the estimate of the errors introduced with them. This is a real difficulty as the Brueckner theory in itself contains approximations not easy to be estimated.

In our treatment, we want to use the variational approach. The essential problem is then how to threat the hard core in the variational calculation.

One procedure would be to replace the repulsive core by an equivalent pseudo-potential of the form

$$4\pi(\hbar^2 a/m)\,\delta(\vec{r}_1 - \vec{r}_2) \tag{1}$$

discussed by Huang and Yang [7] and others [8], [9]. The calculation of the expectation value (1) is essentially equivalent to the Lenz [10]

approximation for kinetic energy. However, the use of the pseudo-potential (1) (constructed for pure hard core two body interaction), plus the attractive part of the potential assumed to be equal to zero for $r < a$, appears to be inadequate for the nuclear saturation problem. Both the kinetic and the potential energies are very sensitive to the detailed shape of the wave function near the surface of the repulsive core at the actual nuclear density. This is not reproduced by the pseudo-potential (1). In the case of an attractive potential which is deep at the surface of the repulsive core (e. g. the Yukawa potential), the representing of the repulsive core with the help of (1) is especially inadequate, whereas in the case of a rather flat attractive part one may argue for such a procedure [11]. For instance, the use of the pseudo-potential (1) with some of the correction terms in the perturbational treatment of the nuclear matter gives in the case of the Gammel-Christian-Thaler potential [12] a much too high binding and density *). Similarly, (1) with the attractive potential [12] gives a much too high binding and density in a variational calculation for the ^{16}O nucleus. Surely, the situation in nuclear matter is essentially different from that in the two-body system. Nevertheless, it is to be noticed that a variational calculation of the binding energy of the deuteron gives, with the help of (1), a binding several times greater than the experimental value (used to fix the parameters of the two-body potential) in the case of the Yukawa potential, although the situation improves when we pass to a flatter potential, e. g. Gaussian.

In this treatment we apply the "Jastrow method", i. e. the simple variational procedure, using a trial function in the form

$$\Psi(1 \ldots A) = \prod_{i<j} f(r_{ij}) \det\{\varphi_k(l)\}. \tag{2}$$

For $\varphi_k(l)$ we use the harmonic oscillator wave functions with one parameter ν. This enables us to apply the well-known technique of the oscillator wave functions ([13], [14], [15]). For f we use the form

$$f(r) = \begin{cases} 0 & \text{for} \quad r \leqslant a \\ 1 - \exp\{-\beta[(r/a)^2 - 1]\} & \text{for} \quad r \geqslant a \end{cases} \tag{3}$$

with the parameter β. We determine the values of the parameters ν, β by minimizing

$$E = \langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle = \mathcal{E} + \Delta + \ldots \tag{4}$$

In (3) we apply the cluster-development ([16], [17], and [18] where the expressions for $\mathcal{E}$, Δ are given). In determining the optimum values of ν and β we restrict ourselves to the first term $\mathcal{E}$. Afterwards, with these values of ν, β we calculate the next higher cluster term Δ which

*) J. Lascoux and W. Tobocman — private communication.

is of the order Ω_c/Ω as compared with $\mathcal{E}$ (Ω_c = the total correlation volume, Ω = the volume of the system) *).

The results of the calculations, i. e. the optimum values of ν and β and the corresponding values of $\mathcal{E}$ and Δ are given in Table I. $\mathcal{E}$ and Δ

TABLE I

Numerical results (lengths in f, energies in MeV)

a	β	$\nu^{1/2}$	r_0	$\mathcal{E}_T$	$\mathcal{E}_V$	$\mathcal{E}$	Δ_T	Δ_V	Δ	$\mathcal{E}+\Delta$
0.2	0.75	0.9	0.86	984	−1183	−199				
0.6	2	0.55	1.50	651	− 716	− 65	6	−31	−25	−90

with the subscript T or V denote the kinetic or the potential part of $\mathcal{E}$ and Δ, respectively. With the help of our function Ψ it is easy to calculate the mean square radius of the nucleus and so the parameter r_0 in the formula for the nuclear radius. The values of r_0 are given in Table I. We calculated Δ in the case $a = 0.6\,f$ only because the analytical expressions obtained for Δ are rather long, and the additional binding introduced by Δ in the case $a = 0.2\,f$ can not change our conclusions.

The results given in Table I should be compared with the experimental values $r_0 = 1.2\,f$ and $E = -128$ MeV. This suggests the conclusion that a two-body potential fitted to the scattering data, with a repulsive core of the radius somewhere between $0.2\,f$ and $0.6\,f$, is capable of reproducing both binding and density of the ^{16}O nucleus.

Similarly as in (11), the "healing" distance corresponding to the calculated values of β is smaller than the internuclear distance. This indicates how to reconcile the strong nucleon-nucleon interaction with the shell model.

The quantitative side of the problem is far from being satisfactory. The large value of Δ makes it clear that in any attempt at quantitative results one has to consider in the variational treatment at least the cluster terms of order Ω_c/Ω together with the two-body term $\mathcal{E}$. A similar difficulty occurs in the Brueckner theory where some of the omitted terms of the second and higher orders in the transition matrix may cause a serious change of the calculated binding energy ([19], [20], [21]).

Since the binding energy is the small difference between a large potential and kinetic parts, the resulting binding is sensitive to the details of the two-body potential (e. g. the interaction in the P state) although they are of small relative importance for the potential part alone. Similar sensitivity was observed in the calculation based on the

*) In calculating Δ an approximation was applied. The corresponding error is expected to be of about 5%.

Brueckner theory *). This makes the situation still more difficult as, at the moment, the details of the two-body interaction are not known.

It is well known that a variational method can give good value for the energy with a poor approximation to the wave function. Nevertheless, we feel that we should supply a foundation for our choice of the correlation function f. This is attempted in the Appendix, where it is shown that, if we put our wave function (2) into the Schroedinger equation, we obtain in the lowest order in Ω_c/Ω an equation for f which only slightly differs from the Bethe-Goldstone equation [22]. But the Bethe-Goldstone equation gives correlations which are essentially negative [11]. This substantiates our form of f varying monatomically from zero at $r = a$ to unity at larger r.

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Appendix

Let us consider the problem of the ground state of A interacting fermions with the hamiltonian

$$H = \sum_{i=1}^{A} t_i + \tfrac{1}{2} \sum_{i=1}^{A} \sum_{j=1}^{A} V(ij)\,, \tag{1A}$$

where t_i is the kinetic energy operator of the i-th particle and $V(ij)$ is the interaction between the i-th and j-th particle. For simplicity we do not consider the spin variables, i. e. we assume that the particles are spinless.

The starting point of the "Jastrow method" [16] of dealing with this problem is the assumption that the wave function of the system has the form

$$\Psi = \prod_{1 \leqslant i < j \leqslant A} f(ij) \sum_{P} \epsilon_P \prod_{n=1}^{A} \varphi_{p_n}(n)\,, \tag{2A}$$

where $\sum_P$ is extended over all the permutations $P = \begin{pmatrix} 1 & \dots & n & \dots & A \\ p_1 & \dots & p_n & \dots & p_A \end{pmatrix}$ with the corresponding signs ϵ_P. The letters i, j, n denoting the arguments of the functions indicate the co-ordinates of the i-th, j-th, n-th particle respectively.

We restrict ourselves here to consider the case of infinite medium (nuclear matter), i. e. we assume for the independent particle wave functions φ_l the form

$$\varphi_l(n) = \Omega^{-1/2} e^{i\vec{k}_l \vec{r}_n}, \tag{3A}$$

*) J. L. Gammel — private communication.

where Ω is the volume of the system. The wave vectors occupy all the states in the Fermi sea:

$$0 \leqslant k_l \leqslant k_F = (6\pi^2)^{1/3} \varrho^{1/3}, \tag{4A}$$

where $\varrho = A/\Omega$ is the density of the system.

The correlation function f is assumed to satisfy the symmetry condition

$$f(ij) = f(ji), \tag{5A}$$

the boundary condition in infinity

$$f(ij) \to 1 \quad \text{for} \quad r_{ij} \to \infty, \tag{6A}$$

and in the case of the presence of a repulsive core of the radius a in $V(ij)$ the boundary condition

$$f(ij) = 0 \quad \text{for} \quad r_{ij} = a. \tag{7A}$$

The region in which $f(ij)$ differs appreciably from unity will be denoted by Ω_c/A, where Ω_c is the total correlation volume. The assumption underlying the Jastrow method is that

$$\Omega_c \ll \Omega. \tag{8A}$$

The functions φ_l and f are determined in the Jastrow method from the variational principle

$$\delta \langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle = 0. \tag{9A}$$

Usually one solves the Eq. (9A) by assuming certain form of φ_l and f containing some parameters and by minimizing $\langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle$ with respect to these parameters. In the case of infinite medium the only parameter is φ_l in the density ϱ. In calculating $\langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle$ one considers the lowest terms in Ω_c/Ω, i. e. the lowest terms in the cluster development.

Here we will follow a different way. We want to answer the question: what is the shape of the correlation function f for a given density ϱ. With this aim in view we insert the wave function (2A) into the Schroedinger equation

$$(H - E)\Psi = 0 \tag{10A}$$

and try to find the equation for f which follows from Eq. (10A), systematically retaining the lowest terms in Ω_c/Ω only. We proceed similarly to Weisskopf and his collaborators [11] in their derivation of the Bethe-Goldstone equation. Namely, we multiply the left side of Eq. (10A) by $\prod_{n=3}^{A} \varphi_n^*(n)$ and integrate over the co-ordinates of the particles $3, 4, \ldots A$:

$$\int d\tau_3 \ldots d\tau_A \prod_{n=3}^{A} \varphi_n^*(n) \Big\{ \sum_{l=1}^{A} t_l + \tfrac{1}{2} \sum_{l=1}^{A} \sum_{s=1}^{A} V(ls) - E \Big\} \Psi = 0. \tag{11A}$$

This is the equation sought for $f(12)$ which we put in a simple form by neglecting all the terms of the order Ω_c/Ω.

Let us first consider the terms with $\sum t_l - E$:

$$(12A)\quad \int d\tau_3 \dots d\tau_A \prod_{n=3}^{A} \varphi_n^*(n) \Big\{ \sum_{l=1}^{A} t_l - E \Big\} \Psi =$$

$$= \Big\{ t_1 + t_2 - \Big[E - \sum_{l=3}^{A} e_l \Big] \Big\} f(12) \sum_P \epsilon_P \varphi_{p_1}(1) \varphi_{p_2}(2) \times$$

$$\times \int d\tau_3 \dots d\tau_A \prod_{n=3}^{A} \varphi_n^*(n) \prod_{\substack{1 \leqslant i < j \leqslant A \\ ij \neq 12}} f(ij) \prod_{m=3}^{A} \varphi_{p_m}(m) ,$$

where $e_l = \hbar^2 k_l^2 / 2m$ is the kinetic energy of the l-th particle. To obtain in (12A) the lowest term in Ω_c/Ω we put all the f functions under the integral

$$(13A)\qquad f(ij) = 1 \quad \text{for} \quad ij \neq 12$$

and get

$$(14A)\qquad \Big\{ t_1 + t_2 - \Big[E - \sum_{l=3}^{A} e_l \Big] \Big\} \Psi_{12}(12),$$

where we introduce the notations:

$$(15A)\qquad \Psi_{mn}(ls) = f(ls) \Phi_{mn}(ls) ,$$

$$(16A)\qquad \Phi_{mn}(ls) = \varphi_m(l) \varphi_n(s) - \varphi_n(l) \varphi_m(s) .$$

To calculate the term in (11A) containing V we split it into four parts:

$$(17A)\quad \tfrac{1}{2} \sum_{l=1}^{A} \sum_{s=1}^{A} V(ls) = V(12) + \sum_{l=3}^{A} V(1l) + \sum_{l=3}^{A} V(l2) + \tfrac{1}{2} \sum_{l=3}^{A} \sum_{s=3}^{A} V(ls) .$$

The calculation of the first term $V(12)$ follows exactly the calculation of the term $\sum t_l - E$, and with the approximation (13A) we get

$$(18A)\qquad V(12) \Psi_{12}(12)$$

The second term gives:

$$(19A)\quad \int d\tau_3 \dots d\tau_A \prod_{n=3}^{A} \varphi_n^*(n) \sum_{l=3}^{A} V(1l) f(12) \sum_P \epsilon_P \varphi_{p_1}(1) \varphi_{p_2}(2) f(1l) \times$$

$$\times \prod_{2 \leqslant i < j \leqslant A} f(ij) \prod_{m=3}^{A} \varphi_{p_m}(m) .$$

Here, we have to keep $f(1l)$ because of the occurrence of $V(1l)$, but the remaining $f-s$ may be in the lowest order of Ω_c/Ω approximated by:

$$(20A)\qquad f(ij) = 1 \quad \text{for} \quad 2 \leqslant i < j \leqslant A .$$

In view of the orthogonality of the functions φ_n we are then left with the permutations $P = \begin{pmatrix} 1 & 2 & 3 \dots l \dots A \\ p_1 & p_2 & 3 \dots p_l \dots A \end{pmatrix}$ only, which give:

$$\text{(21A)} \quad \sum_{l=3}^{A} \Big\{ -f(12)\varphi_l(2) \int d\tau_l \varphi_l^*(l) V(1l) \Psi_{12}(12) + f(12)\varphi_1(2) \times$$

$$\times \int d\tau_l \varphi_l^*(l) V(1l) \Psi_{l2}(1l) + f(12)\varphi_2(2) \int d\tau_l \varphi_l^*(l) V(1l) \Psi_{1l}(1l) \Big\}$$

$$= \sum_{l=3}^{A} \sum_{s=1}^{\infty} \Big\{ -f(12)\varphi_s(1)\varphi_l(2)(sl|V|\Psi_{12}) + f(12)\varphi_s(1)\varphi_1(2) \times$$

$$\times (sl|V|\Psi_{l2}) + f(12)\varphi_s(1)\varphi_s(2)(sl|V|\Psi_{1l}) \Big\},$$

where we introduce the following notation:

$$\text{(22A)} \qquad (sl|V|\Psi_{mn}) = \int d\tau_1 \int d\tau_2 \varphi_s^*(1)\varphi_l^*(2) V(12) \Psi_{mn}(12) .$$

In the last step in (21A) we used the fact that the functions φ_s form a complete set.

The procedure with the third term of (17A) is exactly the same. Combining together the second and third terms of (17A) we get

$$\text{(23A)} \quad \int d\tau_3 \dots d\tau_A \prod_{n=3}^{A} \varphi_n^*(n) \sum_{l=3}^{A} [V(1l) + V(l2)] \Psi =$$

$$= \sum_{l=3}^{A} \sum_{s=1}^{\infty} \{ -\Psi_{sl}(12)(sl|V|\Psi_{12}) + \Psi_{1s}(12)(ls|V|\Psi_{l2}) + \Psi_{s2}(sl|V|\Psi_{1l}) \} .$$

The procedure with the last term of (17A) is similar. We make the approximation

$$\text{(24A)} \qquad f(ij) = 1 \quad \text{for} \quad (ij) \neq (12), (ls) ,$$

and we are left with the permutations $P = \begin{pmatrix} 1 & 2 & 3 \dots l \dots s \dots A \\ p_1 & p_2 & 3 \dots p_l \dots p_s \dots A \end{pmatrix}$ only, which give

$$\text{(25A)} \quad \int d\tau_3 \dots d\tau_A \prod_{n=3}^{A} \varphi_n^*(n) \tfrac{1}{2} \sum_{l=3}^{A} \sum_{s=3}^{A} V(ls) \Psi = \tfrac{1}{2} \sum_{l=3}^{A} \sum_{s=3}^{A} \{ \Psi_{12}(12)(sl|V|\Psi_{sl}) +$$

$$+ \Psi_{sl}(12)(sl|V|\Psi_{12}) - 2\Psi_{1s}(12)(ls|V|\Psi_{l2}) - 2\Psi_{s2}(12)(sl|V|\Psi_{1l}) \} .$$

We now take advantage of the fact that we deal with an infinite medium, i. e. that the functions φ_l are plane waves *).

*) It should be noted that, in the derivation of the cluster development, one always uses plane waves to be able to estimate the order of magnitude of the different terms with the help of the momentum conservation law.

Then the momentum conservation law demands that $s = 1$, $s = 2$ in $(sl|V|\Psi_{11})$; $(ls|V|\Psi_{l2})$ respectively. This gives us, after combining together Eqs. (14A), (18A), (23A), (25A), the equation

$$\{t_1 + t_2 + V(12) - E_{12}\}\Psi_{12}(12) = \tfrac{1}{2}\Big[\sum_{l=3}^{\infty}\sum_{s=3}^{\infty} - \sum_{l=A+1}^{\infty}\sum_{s=A+1}^{\infty}\Big]\Psi_{sl}(12)(sl|V|\Psi_{12}), \tag{26A}$$

where

$$E_{12} = E - \sum_{l=3}^{A} e_l - \tfrac{1}{2}\sum_{\substack{l=1 \\ ls\neq 12,21}}^{A}\sum_{s=1}^{A}(sl|V|\Psi_{sl}). \tag{27A}$$

Multiplying the left side of Eq. (26A) by $\varphi_1^*(1)\varphi_2^*(2)$, integrating over the co-ordinates of the 1st and 2nd particle, and neglecting terms of order Ω_c/Ω we get:

$$E_{12} = e_1 + e_2 + (12|V|\Psi_{12}), \tag{28A}$$

hence

$$E = \tfrac{1}{2}\sum_{s=1}^{A}\sum_{l=1}^{A} E_{ls}, \tag{29A}$$

where E_{ls} is defined like E_{12} in (28A). With the help of (28A) and the momentum conservation law we may write (26A) in the form:

$$\{t_1 + t_2 + V(12) - e_1 - e_2\}\Psi_{12}(12) = \tfrac{1}{2}\Big[\sum_{l=1}^{\infty}\sum_{s=1}^{\infty} - \sum_{l=A+1}^{\infty}\sum_{s=A+1}^{\infty}\Big]\Psi_{sl}(12)(sl|V|\Psi_{12}). \tag{30A}$$

Eq. (30A) is an equation for Ψ_{12}, but, of course, identical equations hold for any Ψ_{ls}. Hence, we have together $A(A-1)/2$ equations for one function f, since we fixed the independent particle wave functions φ_l. This shows that, if one tries to satisfy the Schroedinger equation with the Jastrow type wave function, one has to introduce state dependent correlation functions f_{ls}. Although the calculations with such state dependent correlation functions are essentially more complicated, we hope that equations of the type (30A) should still be obtained.

Eq. (30A) differs from the Bethe-Goldstone equation (22) only by the occurrence of $\Psi_{sl}(12)$ on its right side instead of $\Phi_{sl}(12)$ occurring in the Bethe-Goldstone equation. The equations (30A) have the property that both sides vanish inside the repulsive core. For great distance r_{12}, Eq. (30A) is of course identical with the Bethe-Goldstone equation.

This brief report on papers [1], [2] with new arguments given in the Appendix was prepared by the author at the Department of Mathematical Physics, University of Birmingham, England, while on leave of absence from the Institute of Theoretical Physics, Warsaw University.

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On the Motion of Rotating Bodies in the General Theory of Relativity

by

W. TULCZYJEW

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It is the aim of the present paper to derive general relativistic equations for the motion of rotating bodies and to compute the relativistic secular corrections for Newtonian motion due to the rotation of bodies.

1. The motion of a system of two bodies of comparable masses, as constituted e. g. by a double star, was first considered in general relativity theory in 1938 by Einstein, Infeld and Hoffman (EIH) [1], who, from the equations of the gravitational field, derived approximate equations of motion of two spherically symmetric, non-rotating bodies represented by field singularities of the single pole type. The method of approximation as adapted to the investigation of slow motion yielded, in the first approximation, the Newtonian equations. The corrections obtained in the next step of the approximation, as shown by Robertson [2], lead to the motion of the periastron. The EIH method underwent numerous simplifications and developments [3], [4], [5], whilst a different method [6], [7], based on a description of the bodies by means of continuous energy-momentum density, was being elaborated simultaneously. Attempts were also made to account for the effect of the rotation in post-Newtonian approximation. Equations of motion of rotating bodies were proposed by Fok [8], Haywood [9], and Ryabushko [10], none of whom, however, gave the equations in integrated form. The effect of rotation upon motion is known only for the case of a test particle moving in the field of a rotating body [11].

2. For deriving the equations of motion, Infeld's method is applied [5], whereby the bodies are considered to be represented by singularities of the gravitational field, and are described by the "singular energy-momentum density" constructed with Dirac's δ-function *). Sphe-

*) In order to avoid divergences, the modified function as introduced by Infeld and Plebański [12], [13] should be used in place of the usual, three-dimensional Dirac δ

rically symmetric non-rotating bodies are treated as singularities of the most simple type and accounted for by the density:

$$T^{\alpha\beta} = \sum_{P} \overset{P}{t}{}^{\alpha\beta}(t)\,\delta\big(x^r - \overset{P}{\xi}{}^r(t)\big)\,, \tag{2.1}$$

wherein the $\overset{P}{\xi}{}^r$ are the spatial co-ordinates of the p-th singularity. The Greek indices assume the values $0, 1, 2, 3$, the Latin indices $-1, 2, 3$. To describe rotating bodies, it is necessary to introduce the derivatives of the δ-function:

$$T^{\alpha\beta} = \sum_{P} \big(\overset{P}{t}{}^{\alpha\beta}(t)\,\overset{P}{\delta} - \overset{P}{t}{}^{r\alpha\beta}(t)\,\overset{P}{\delta}{}_{,r}\big); \qquad \overset{P}{\delta} = \delta(x^r - \overset{P}{\xi}{}^r)\,. \tag{2.2}$$

The equations of motion represent the conditions of integrability of the field equations, and are of the general form:

$$\begin{gathered}
\int\limits_{\Omega(\overset{P}{\xi}{}^r)} T^{\alpha\beta}{}_{;\beta}\, d_{(3)}x = 0\,, \\
\int\limits_{\Omega(\overset{P}{\xi}{}^r)} (x^r - \xi^r) T^{\alpha\beta}{}_{;\beta}\, d_{(3)}x = 0\,, \\
\int\limits_{\Omega(\overset{P}{\xi}{}^r)} (x^r - \xi^r)(x^s - \xi^s) T^{\alpha\beta}{}_{;\beta}\, d_{(3)}x = 0\,,
\end{gathered} \tag{2.3}$$

with $\Omega(\overset{P}{\xi}{}^r)$ denoting the three-dimensional neighbourhood of the p-th singularity, and the semi-colon standing for the covariant derivative. The world lines of singularities should be related to the centres of mass of the bodies by the conditions:

$$\overset{P}{t}{}^{r00} = 0\,. \tag{2.4}$$

With these conditions, Eqs. (2.3) determine the motion of the bodies uniquely, without any additional assumptions whatsoever on the structure of the coefficients $\overset{p}{t}{}^{\alpha\beta}$ and $\overset{P}{t}{}^{r\alpha\beta}$. On applying the method of approximation to the field equations

$$R^{\alpha\beta} = -8\pi(T^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} g_{\varrho\sigma} T^{\varrho\sigma}) \tag{2.5}$$

and to Eqs. (2.3), the equations of motion containing only parameters characterizing the positions and rotation of the bodies are obtained. The equations will be given without the proof for two bodies, in post-Newtonian approximation *):

*) The equations are derived by a generalization of the method given in a paper by Infeld [5] for non-rotating bodies. The calculations will be published in full detail separately in Acta Phys. Polon.

$$(2.6)\quad \overset{1}{m}\ddot{\eta}^m - \overset{1}{m}\overset{2}{m}\left(\frac{1}{r}\right)_{,\eta^m} = \overset{1}{m}\overset{2}{m}\left\{\left[\dot{\eta}^s\dot{\eta}^s + \frac{3}{2}\dot{\zeta}^s\dot{\zeta}^s - 4\dot{\eta}^s\dot{\zeta}^s - 4\frac{\overset{2}{m}}{r} - 5\frac{\overset{1}{m}}{r}\right]\left(\frac{1}{r}\right)_{,\eta^m} + \right.$$

$$+ [4\dot{\eta}^s(\dot{\zeta}^m - \dot{\eta}^m) + 3\dot{\eta}^m\dot{\zeta}^s - 4\dot{\zeta}^m\dot{\zeta}^s]\left(\frac{1}{r}\right)_{,\eta^s} + \frac{1}{2}(r)_{,\eta^r\eta^s\eta^m}\dot{\zeta}^r\dot{\zeta}^s\Big\} +$$

$$+ [\overset{2}{m}\overset{1}{S}{}^{rs}(\dot{\eta}^s - 2\dot{\zeta}^s) + \overset{1}{m}\overset{2}{S}{}^{rs}(2\dot{\eta}^s - \dot{\zeta}^s)]\left(\frac{1}{r}\right)_{,\eta^r\eta^m} +$$

$$+ [2\overset{2}{m}\overset{1}{S}{}^{rm}(\dot{\eta}^s - \dot{\zeta}^s) + 2\overset{1}{m}\overset{2}{S}{}^{mr}(\dot{\eta}^s - \dot{\zeta}^s)]\left(\frac{1}{r}\right)_{,\eta^r\eta^s},$$

$$(2.7)\quad \frac{d\overset{1}{m}}{dt} = 0,$$

$$(2.8)\quad \frac{d\overset{1}{S}{}^{rs}}{dt} = 0;$$

$\eta^r = \overset{1}{\xi}{}^r$ and $\zeta^r = \overset{2}{\xi}{}^r$ are the co-ordinates of the bodies, $r = [(\eta^s - \zeta^s)(\eta^s - \zeta^s)]^{1/2}$, the dot stands for time derivatives, $\overset{1}{m}$ and $\overset{2}{m}$ denote the Newtonian masses of the bodies, and $\overset{P}{S}{}^{rs} = -\overset{P}{S}{}^{sr}$ represent the internal angular momenta. The equations of the second body are obtained if $\eta^r, \zeta^r, \overset{1}{m}, \overset{2}{m}, \overset{1}{S}{}^{rs}, \overset{2}{S}{}^{rs}$ are replaced by $\zeta^r, \eta^r, \overset{2}{m}, \overset{1}{m}, \overset{2}{S}{}^{rs}, \overset{1}{S}{}^{rs}$, respectively. The left hand side of Eqs. (2.6) contains the Newtonian terms, whilst the right hand side contains the known EIH terms and the terms accounting for rotation. It is the significance of these latter terms for the path of relative motion of the bodies that is the central point of interest in the present investigation.

3. Eqs. (2.6) will now be integrated, neglecting the EIH terms and treating the terms related to rotation as small corrections to the Newtonian equations. In integrating, it is most convenient to use the Lagrangian. It is easily verifiable that the latter, which is of the form:

$$(3.1)\quad L = \tfrac{1}{2}\overset{1}{m}\dot{\eta}^s\dot{\eta}^s + \tfrac{1}{2}\overset{2}{m}\dot{\zeta}^s\dot{\zeta}^s + \frac{\overset{1}{m}\overset{2}{m}}{r} -$$

$$- \frac{\overset{1}{m}}{r^3}\overset{2}{S}{}^{rs}(2\dot{\eta}^s - \dot{\zeta}^s)(\eta^r - \zeta^r) - \frac{\overset{2}{m}}{r^3}\overset{1}{S}{}^{rs}(\dot{\eta}^s - 2\dot{\zeta}^s)(\eta^r - \zeta^r)$$

leads to Eqs. (2.6) and to analogous equations for the second body by variation of η^r and ζ^r, if the EIH terms are neglected. With the following notation:

$$x^r = \eta^r - \zeta^r$$

$$X^r = \frac{1}{M}(\overset{1}{m}\eta^r + \overset{2}{m}\zeta^r)$$

$$M = \overset{1}{m} + \overset{2}{m}$$

(3.2)

$$\mu = \frac{\overset{1}{m}\overset{2}{m}}{M}$$

$$Ms^{rs} = \left(1 + \frac{1}{2}\frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{S}{}^{rs} + \left(1 + \frac{1}{2}\frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{S}{}^{rs}$$

$$M\delta s^{rs} = \left(1 + \frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{S}{}^{rs} - \left(1 + \frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{S}{}^{rs}$$

the Lagrangian (3.1) is rewritten in three-dimensional vector notation as follows:

$$L = \frac{1}{2}MV^2 + \frac{\mu M}{r^3}\vec{V}(\vec{r}\times\delta\vec{s}) + \mu\left[\frac{1}{2}v^2 + \frac{M}{r} + 2\frac{M}{r^3}\vec{v}(\vec{r}\times\vec{s})\right]. \tag{3.3}$$

The vectors $\vec{r}$ and $\vec{R}$ have the components x^r and X^r, $\vec{v} = \dot{\vec{r}}$, $\vec{V} = \dot{\vec{R}}$, the vector $\vec{s}$ has the components $s^r = \varepsilon^{rst}S^{st}$ ($\varepsilon^{rst} = \varepsilon^{trs} = \varepsilon^{str} = -\varepsilon^{srt} = -\varepsilon^{tsr} = -\varepsilon^{rts}$, $\varepsilon^{123} = 1$), and similarly the vector $\delta\vec{s}$. The Lagrangian (3.1) is invariant with respect to the displacements

$$\vec{R}' = \vec{R} + \delta\vec{R}, \tag{3.4}$$

where $\delta\vec{R}$ is an arbitrary constant vector. By Noether's theorem [14], certain conservation principles are attached to the invariance of the Lagrangian with regard to infinitesimal transformations *). Invariance with regard to displacement leads to the following integrals of momentum:

$$M\vec{V} + \frac{\mu M}{r^3}(\vec{r}\times\delta\vec{s}) = \vec{P}. \tag{3.5}$$

which, by applying the Newtonian equations of relative motion, yield:

$$M\vec{R} = \mu(\vec{v}\times\delta\vec{s}) + \vec{P}t + \vec{K}, \qquad \vec{K} = \overrightarrow{\text{const}}, \tag{3.6}$$

i. e. rotation of the bodies leads to small oscillations of the Newtonian centre of mass about the centre of inertia moving uniformly. Restricting the investigation to reference systems attached to the centre of inertia ($\vec{K} = 0$, $\vec{P} = 0$), the terms containing $\vec{V}$ may be neglected in the La-

*) An application of Noether's theorem for deriving conservation principles in Newtonian mechanics is to be found in a paper by Hill [15].

grangian as small terms of higher order. On omitting the constant factor μ, the following Lagrangian of relative motion is obtained:

$$L' = \frac{1}{2}v^2 + \frac{M}{r} + 2\frac{M}{r^3}\vec{v}(\vec{r}\times\vec{s}), \tag{3.7}$$

the latter being invariant with regard to displacements of time:

$$t' = t + \delta t. \tag{3.8}$$

Hence, the following integral of the energy is obtained:

$$\frac{1}{2}v^2 - \frac{M}{r} = E. \tag{3.9}$$

Invariance with respect to rotations:

$$\begin{aligned} \vec{r}' &= \vec{r} + \vec{\varepsilon}\times\vec{r}, \\ \vec{s}' &= \vec{s} + \vec{\varepsilon}\times\vec{s} \end{aligned} \tag{3.10}$$

yields

$$\dot{\vec{J}} = 2\frac{M}{r^3}\vec{s}\times(\vec{r}\times\vec{v}) = 2\frac{M}{r^3}\vec{s}\times\vec{J}, \tag{3.11}$$

wherein

$$\vec{J} = \vec{r}\times\vec{v} + 2\frac{M}{r^3}\vec{r}\times(\vec{r}\times\vec{s}). \tag{3.12}$$

Some typical, special cases of motion will now be considered. First, it will be assumed that motion is bounded and takes place in the plane perpendicular to the vector $\vec{s}$. Here, Eqs. (3.11) and (3.12) yield:

$$\vec{J} = \vec{r}\times\vec{v} - 2\frac{M}{r}\vec{s} = \overrightarrow{\text{const}}. \tag{3.13}$$

The introduction of the polar co-ordinates r, φ in the plane of motion yields

$$J = r^2\dot{\varphi} - 2\frac{Ms}{r}. \tag{3.14}$$

The right hand side of the identity

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = v^2(r^2\dot{\varphi})^{-2}; \qquad u = \frac{1}{r}, \tag{3.15}$$

may be represented by (3.14) and (3.9) as a polynomial in the variable u. Hence,

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = J^{-2}\left[2E + \left(2M - 8\frac{s}{J}EM\right)u - 8\frac{s}{J}M^2u^2\right]. \tag{3.16}$$

The solution of the latter equation in the Newtonian approximation is given by Kepler's ellipse:

$$u = p^{-1}\left(1 + \varepsilon\cos(\varphi - \varphi_0)\right), \qquad p = \frac{J^2}{M}, \qquad \varepsilon = \left(1 + \frac{2EJ^2}{M^2}\right)^{1/2}. \tag{3.17}$$

The next approximation yields a rotating ellipse with the following displacement of the periastron:

$$-8\pi\frac{Ms}{p\bar{J}} \tag{3.18}$$

for each period of the classical motion. Another case is that of classical motion along the circular path $r = a$. Eq. (3.11) now yields:

$$\dot{\vec{J}} = 2\frac{M}{a^3}\vec{s}\times\vec{J}\,. \tag{3.19}$$

Thus, the vector $\vec{J}$ precesses about the constant vector $\vec{s}$ with angular velocity

$$2\frac{M}{a^3}\vec{s}\,. \tag{3.20}$$

The latter angular velocity is also that of the line of equinoxes, i. e. the line of intersection of the plane of motion, which is perpendicular to the vector $\vec{J}$, and the plane perpendicular to the vector $\vec{s}$. The above results are in agreement with those of Lense and Thirring [11] for test particles.

4. In the previous section, the motion of bodies with regard to a reference system was investigated. The motion should now be related to that observed by a distant observer through the intermediary of light beams. This requires that the gravitational field in the space surrounding the bodies be given with the degree of approximation necessary for writing the approximate equations of the light rays. In the reference system, wherein Eqs. (2.6) hold, the gravitational field is of the form:

$$\begin{aligned} &g_{00} = 1-2\overset{1}{m}\overset{1}{/r}-2\overset{2}{m}\overset{2}{/r}\,,\\ &g_{0n} = 0\,,\\ &g_{mn} = -\delta_{mn}(1+2\overset{1}{m}\overset{1}{/r}+2\overset{2}{m}\overset{2}{/r});\\ &\overset{P}{r} = [(x^r-\overset{P}{\xi}{}^r)(x^r-\overset{P}{\xi}{}^r)]^{1/2}\,. \end{aligned} \tag{4.1}$$

It should be noted that the gravitational field $g_{\alpha\beta}$ in this reference system takes the Galilean values $\eta_{\alpha\beta}$ ($\eta_{00} = 1$, $\eta_{rs} = -\delta_{rs}$, $\eta_{0r} = 0$) at space infinity and is time dependent only through the positions of the bodies.

These properties of the gravitational field will now be used to prove that the effects of the previous section coincide with those perceived by a distant observer. Motion as observed for short intervals of time is described with sufficient accuracy by the Newtonian equations, whose interpretation presents no difficulties, since, in the Newtonian approxi-

mation, the light signals travel along straight lines. Post-Newtonian corrections, as already proved, result in slow, periodical variations of the position of the Newtonian path. The periods of these variations are the only quantities of interest in the problem. Periodicity of motion with regard to the reference system used involves periodicity of the time variations of the gravitational field. It may thus be concluded, even without investigating the path of the light rays in detail, that the motion as observed from a great distance will also be periodical. The behaviour of the gravitational field at infinity guarantees that the period with regard to the reference system shall be equal to the period observed. This yields the proof that the effects derived in the previous section are really those observed from a great distance.

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EXPERIMENTAL PHYSICS

Excitation Energy Transfer from Dye Molecules in the Metastable State

by

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The aim of the investigations presented in this paper is to provide answers to the following questions:

1) Is the fundamental degree of polarization of one and the same band appearing in fluorescence and phosphorescence the same or different?

2) Is there or is there not excitation energy transfer from molecules in the metastable state?

The results published hitherto seem to indicate that the degree of polarization of phosphorescence is lower than that of the polarization of fluorescence. The differences in the degrees of polarization suggest that phosphorescence light undergoes self-depolarization to a greater extent than fluorescence light. This last fact, in turn, would seem to indicate that excitation energy may also be transferred, as suggested by Laffitte [8], from molecules excited to the metastable state.

In considering the problem of energy transfer, Vavilov [9] concludes that no such transfer can take place from the metastable state M. According to this latter author, the effect considered is possible only from the F level. Yermolayev and Svyeshnikov [5] also represent the opinion that the molecule in the metastable state is not able to transfer the excitation energy to other molecules. Since, however, the existence of self-depolarization within the long-wave fluorescence band ($M \rightarrow N$) has been experimentally proved, they explain this depolarization by assuming the molecule to exist in a hypothetical transient "unsteady" state from which energy transfer may take place. This interpretation was suggested to them by the fact that, according to their measurements, the time during which the molecule exists in the M state has no effect upon the degree of polarization.

To contribute towards the solution of these problems, the dependence of the degree of polarization of fluorescence and phosphorescence

on the concentration of the dye molecules was investigated for acridine yellow in boric acid.

The measurements were made using a device described in detail elsewhere [4]. The results obtained are to be found in Table I, wherefrom it will be seen that the degree of polarization of the "pure" phosphorescence light as well as that of the mixture of phosphorescence and fluorescence depend on the concentration of the dye. To determine the polarization of "pure" fluorescence, it is necessary to measure the

TABLE I

Results of measurements of emission anisotropy of luminescence of acridine yellow in boric acid solution *versus* the concentration, compared to theoretical results computed by formula.

c (g/g)	n (cm^{-3})	ν_P	ν_F	r_F exp	r_F	r_P exp	r_P	η_P/η_F
5×10^{-6}	19×10^{15}	0.01	0.003	0.23	0.23	0.23	0.23	<0.017
5.6×10^{-5}	22×10^{16}	0.14	0.04	0.23	0.23	0.22	0.22	0.034
3×10^{-4}	12×10^{17}	0.78	0.24	0.203	0.207	0.157	0.16	0.085
5.9×10^{-4}	23×10^{17}	1.49	0.46	0.198	0.194	0.12	0.12	0.119
1×10^{-3}	4×10^{18}	2.60	0.80	0.172	0.178	0.11	0.092	0.07

c — concentration of dye molecules, in g/g,
n — number of dye molecules per cm^3,
ν_P, ν_F — ν for phosphorescence and fluorescence, respectively,
r_F^{exp} — fluorescence emission anisotropy as experimentally determined,
r_F — fluorescence emission anisotropy computed theoretically from Eq. (8),
r_P^{exp}, r_P — respective phosphorescence emission anisotropies,
$\nu = nv$, where v is the volume of the active sphere.

quotient of the phosphorescence and fluorescence yields η_P/η_F and to compute the emission anisotropy r from the expression [2]:

$$r_F = r_{P+F} + \frac{\eta_P}{\eta_F}(r_{P+F} - r_P), \tag{1}$$

the relation between the quantity r and the degree of polarization P being given in a paper by A. Jabłoński [7]. Here, r_F denotes the anisotropy of fluorescence emission, r_P – that of phosphorescence emission, and r_{P+F} – that of the emission of phosphorescence and fluorescence jointly. The quotient η_P/η_F was measured with a device described in [1]. In all measurements, the same crossed filters and the same source of light were used. The resulting dependence of η_P/η_F on the concentration of the dye molecules is shown in Fig. 1. The dependence of the phosphorescence emission anisotropy, and that of the fluorescence anisotropy computed by means of Eq. (1) on the concentration of the dye molecules is shown in Fig. 2.

The shape of the dependence of the emission anisotropies of the fluorescence and phosphorescence on the concentration of the dye molecules may be explained on the basis of A. Jabłoński's theory [6], [7].

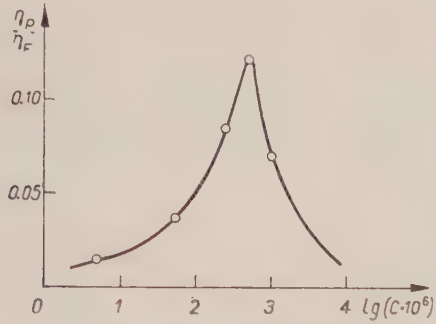

Fig. 1

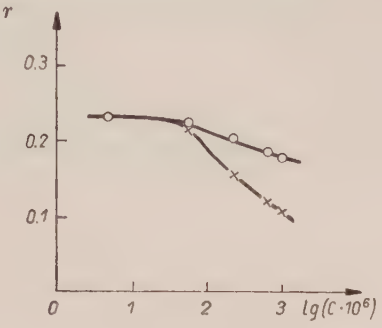

Fig. 2

Fig. 1. The ratio of phosphorescence yield to the fluorescence yield *versus* concentration of dye molecules
∘ — experimental points

Fig. 2. Fluorescence and phosphorescence emission anisotropies *versus* the concentration
—∘— — fluorescence emission anisotropy
—×— — phosphorescence emission anisotropy

According to the theory, the anisotropy of the phosphorescence emission may be written in the form

$$r_P = \frac{\sum_{k=1}^{\infty} r_k \int_0^{\infty} I_k(t)\,dt}{\sum_{k=1}^{\infty} \int_0^{\infty} I_k(t)\,dt}, \tag{2}$$

where $r_k = \frac{r_0}{k}$ denotes the emission anisotropy of group k for $t \to \infty$, and

$$\int_0^{\infty} I_k(t)\,dt = \frac{I_{k_0}}{\Gamma_k} \tag{3}$$

with

$$I_{k_0} = B\,\frac{\gamma_k^2}{\Gamma_k}\,\frac{\nu^{k-1}}{(k-1)!}\left[1 - e^{-\Gamma_k t_0}\right]. \tag{4}$$

In the above expressions, the index k denotes the number of luminescent molecules forming the centre belonging to group k, r_0 is the fundamental emission anisotropy, $I_k(t)$ — the phosphorescence intensity emitted by the k-th group of centres, $I_{k_0} = I_k(t)$ at $t = 0$, i. e. at the moment of cessation of excitation, t_0 — duration of excitation, γ_k — the probability per unit time of an electron transition leading to emission of phosphorescence, $\nu = n \cdot v$, with n denoting the number of luminescent

molecules per unit of volume, v — the volume of the active sphere, and B a coefficient of proportionality independent of k.

For prolonged (continuous) excitation, when t_0 attains large values, and in the absence of quenching ($\Gamma_k = \gamma_k$),

$$I_{k_0} = B\gamma_k \frac{\nu^{k-1}}{(k-1)!}. \tag{4'}$$

Substituting (3) and (4') into (2) we have

$$r_P = r_0 \frac{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{k!}}{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!}}, \tag{5}$$

whence

$$r_P = r_0 \frac{1-e^{-\nu}}{\nu}. \tag{6}$$

Eq. (6) is checked against experimental results in Fig. 3 and Table I, ν being computed for a value of $v = 650\times10^{-21}$ cm^3 corresponding to the radius $R = 56$ Å of the active sphere. The small deviations of the experimental points from the theoretical curve, especially within the region of high concentrations, are possibly due to self-quenching, a factor neglected in deriving Eq. (6) *).

For the case of fluorescence, by continuous excitation and the absence of quenching, the emission anisotropy of the k-th group is given by the expression:

$$r_k = r_0 \left(\frac{1}{k} + \frac{k-1}{k} \frac{\gamma_k}{\gamma_k + k\mu}\right). \tag{7}$$

The assumption $\mu \approx \gamma_k$ imposes the volume of the active sphere and leads to the expression **):

$$r_F = r_0 \frac{2(\nu - 1 + e^{-\nu})}{\nu^2}, \tag{8}$$

which, if the radius of the active sphere is chosen as $R = 30$ Å, describes adequately the shape of the self-depolarization curve of the fluorescence (Fig. 4).

It will be seen from Fig. 4 that, for very low concentrations of the dye molecules, luminescence light emission anisotropy is the same for

*) According to Eq. (6), r_P is independent of the time during which the molecule remains on the M level. Eq. (6) should also describe the r dependence on ν for slow fluorescence, for which the above independence was found to hold by Yermolayev and Svyeshnikov [5].

**) Bojarski [3] pointed to the possibility of obtaining this expression in finite form.

fluorescence as for phosphorescence: the fundamental emission anisotropy, and thus also the fundamental polarization, are the same. Hence, we conclude that the fundamental polarization does not depend on whether the transition is a direct one from the F to the N level, or whether it occurs after the molecule has sojourned in the metastable state M. This result is in agreement with Jabłoński's theory. For low concentrations, practically only centres with $k = 1$ exist, and thus the probability of a transfer is zero. As the concentration rises, the number of groups

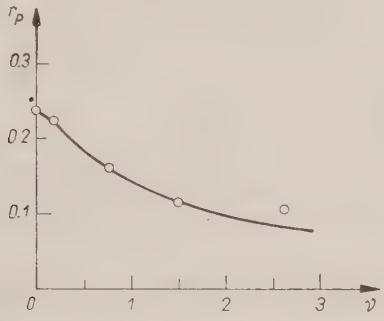

Fig. 3

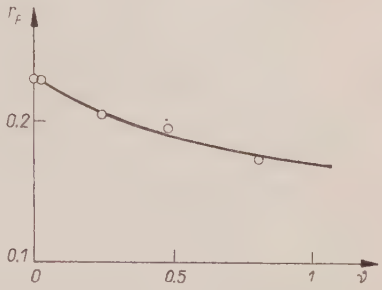

Fig. 4

Fig. 3. Comparison of experimentally determined phosphorescence emission anisotropy and curve representing Eq. (6)
○ — experimental points

Fig. 4. Comparison of experimentally determined fluorescence emission anisotropy and curve representing Eq. (8)
○ — experimental points
— — curve obtained from Eq. (8)

of centres with $k = 2$, $k = 3$, etc grows, and thus also the probability of the transfer of energy leading to depolarization of the luminescence light, involving a decrease of the emission anisotropy. The greater steepness of the phosphorescence self-depolarization curve as compared with the analogous curve for fluorescence indicates the existence of energy transfer from molecules in the metastable state. Results resembling those already described are obtained for fluorescein in boric acid solution.

It remains to be considered whether the energy transfer from the M level proceeds directly, or whether the F level intervenes in the transfer. The following model suggests itself: the $F \rightarrow N$ transition is responsible for the fluorescence; the phosphorescence is produced by $F \rightarrow N$ transitions following one or more sojourns of the molecule in the metastable state. After each $F \rightarrow M$ transition, the molecules would return to the F level, from which energy transfer could then take place. Hence, phosphorescence light from molecules having sojourned many times in the F state would exhibit a higher rate of depolarization than the light from fluorescence. The resulting phosphorescence emission anisotropy would

thus assume an average value below that of the fluorescence emission anisotropy. It would seem, however, that such a model is rendered unacceptable by the way in which the quotient of the yield of phosphorescence and yield of fluorescence depends on the exciting wavelength [1]. This dependence proves that the molecule may go over into the metastable state only from certain higher oscillatory levels of state F. Subsequent to its first transition to the M level, the probability of the molecule resuming a high oscillatory state of the F level by absorbing thermal energy is very low, and hence the probability of multiple transitions from state M to F and back to F is practically zero. Thus, it is to be concluded that energy transfer takes place directly from the M state, and not through the intermediary of the F level.

The authors wish to express their very sincere thanks to Professor A. Jabłoński and Dr. M. Frąckowiak for their numerous discussions and valuable remarks, and also for their interest in the present investigation.

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EXPERIMENTAL PHYSICS

Raman Spectrum of Antimony Trichloride Crystals

by

A. TRAMER

Prestended by A. JABŁOŃSKI on July 8, 1958

The Raman spectrum of antimony trichloride was investigated in the crystal state.

The vibrational spectrum of liquid $SbCl_3$ [1], [2] consists of four frequencies: two symmetrical ones – A: $\nu_1 = 360$ cm^{-1}, $\nu_2 = 160$ cm^{-1} and two doubly degenerate – E: $\nu_{3,4} = 320$ cm^{-1} and $\nu_{5,6} = 135$ cm^{-1}. The free molecule belongs to the symmetry group C_{3v}. In the liquid state all frequencies appear as broad bands because of strong intermolecular interactions (mainly of the dipole – dipole type).

The Raman spectrum of solid $SbCl_3$ was investigated at liquid air temperature by Gupta [3], who observed a splitting of degenerate frequencies into components with intervals of 7 cm^{-1} for $\nu_{3,4}$ and 13 cm^{-1} for $\nu_{5,6}$ and a shift of ν_1 towards the lower frequencies.

More recently, Raskin [4] and Gross [5] published results for antimony trichloride single crystals at room temperature. Their results differ in the assignment of bending frequencies. They demonstrated a shift in valence frequencies, but no splitting of degenerate ones (Table I).

We have reinvestigated the Raman spectrum of $SbCl_3$ single crystals. The crystals were grown in a glass Raman tube by means of zone melting. A "Huet B II" spectrograph was used, with dispersion increased to 8 Å/mm. at about 4400 Å by bringing the prisms out of minimum deviation; the source was a low-pressure quartz mercury lamp with water-cooled liquid electrodes. The spectrum was excited with the blue Hg line (4358 Å) and photographed on "Ramanplatten Agfa" hypersensitive plates.

The resolving power of the apparatus was large enough to resolve the isotope components of the CCl_4 line ν_1 459 cm^{-1} with intervals of about 3 cm^{-1}.

The microphotometer curve of the spectrum is given in Fig. 1, and the frequencies of the lines are compared in Table 1 with the results of other authors. In most cases the values given are the means of the Stokes and anti-Stokes frequencies.

TABLE I

Spectrum of $SbCl_3$				
Liquid	Crystal			
	Gupta (—180° C.)	Raskin	Gross	Author
—	—	—	—	15 v. w.?
—	—	—	—	26 w.
—	—	—	35	36 m.
—	—	—	—	44 w.
—	—	—	50	52 m.
—	—	—	66	64 m. (double? 62, 68)
—	—	—	—	76 v. w.?
—	—	—	—	86 w.
—	—	—	96	95 m.
135	138 151	152	133	139 s. 150 s. $\nu_{5,6}$
160	178	167	152	170.5 s. ν_2
320	310 317	318	317	315 s. 320 v. s. $\nu_{3,4}$
360	338	343	342	343 v. s. ν_1

The six lines of higher frequency may be unambiguously correlated with the intramolecular vibrations of free molecules; the others are the

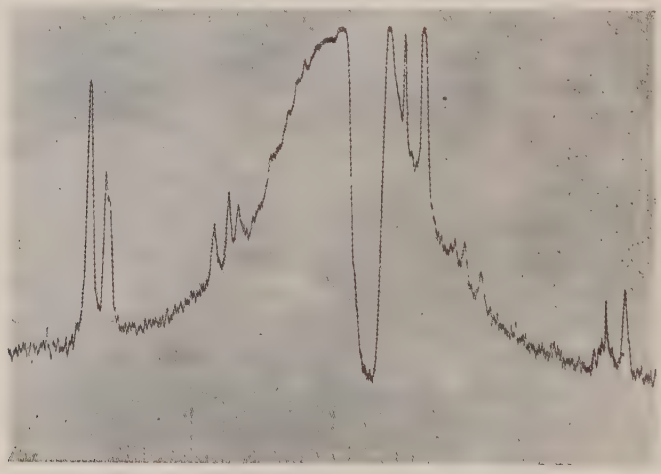

Fig. 1. Raman spectrum of $SbCl_3$ crystal

lattice frequencies. Besides four stronger lines reported by Gross, some weak and very weak (uncertain) lines are present.

Both degenerate frequencies of free molecules split in the crystal state.

The splitting of the bending frequency $\nu_{5,6}$ was large enough to make visible two distinct lines. The frequency difference and intensity ratio remained constant in the two investigated crystal samples.

The structure of the $\nu_{3,4}$ line was only partially resolved. The contour changed slightly from crystal to crystal, perhaps as a result of changes in the width or in the intensity ratio of components, but the frequency difference remained constant (Fig. 2).

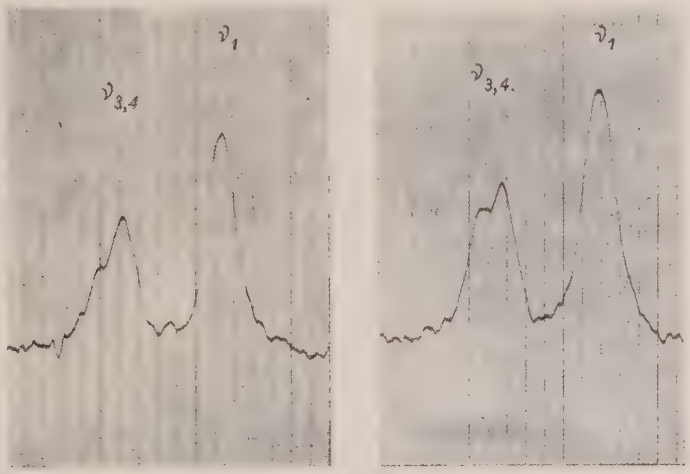

Fig. 2. Valence frequencies in the spectra of different samples

Most likely the degeneracy of vibrations is removed because of the symmetry change, which leads to the splitting of corresponding frequencies.

The crystal structure of $SbCl_3$ is orthorhombic [6], 4 molecules in a unit cell. The symmetry of a unit cell is thus lower than that of the free molecule. The Sb-Cl bonds are not equivalent, the lengths of different bonds are 2.35 ± 0.03 Å and 2.37 ± 0.03 Å.

We have no information on the uniformity of the crystals investigated, or the orientation of crystal axes. It is possible that the differences in the spectra of the samples result from the different orientation of crystals.

Investigations of $SbCl_3$ and related single crystals are in progress.

The author would like to thank Professors A. Jabłoński and T. Skaliński for the interest shown in this work.

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EXPERIMENTAL PHYSICS

Quenching of Photoluminescence of Solutions by Energy Transfer

by

A. JABŁOŃSKI

Presented on August 18, 1958

General remarks

If in a solution there are centres exhibiting different yields of photoluminescence, the observed value of the yield is its mean value $\overline{\eta}$. Thus, if there are $n+1$ distinct groups of centres, whose yields are η_k ($k = 0, 1, \ldots n$),

$$\overline{\eta} = \sum_{k=0}^{n} P_k \eta_k \,, \tag{1}$$

where P_k, fulfilling the condition $\sum_{k=0}^{n} P_k = 1$, is the distribution of the absorbed energy among different groups (cf. [4]).

In the case of a continuous distribution, the sum must be replaced by an integral. Thus, if e. g. the yield $\eta(R_1 \ldots R_z)$ is a function of distances $R_1 \ldots R_z$ of quenchers (quenching molecules) from the excited luminescent molecule, $\overline{\eta}$ is given by

$$\overline{\eta} = \int \ldots \int P(R_1 \ldots R_z)\eta(R_1 \ldots R_z) dR_1 \ldots dR_z \,, \tag{2}$$

where $P(R_1 \ldots R_z)dR_1 \ldots dR_z$ denotes the probability of a configuration, in which the distances of quenchers lie between R_1 and $R_1 + dR_1$, ... R_z and $R_z + dR_z$. The integral has to be taken over the entire range of all R.

In two previous papers [3], [4] $\overline{\eta}$ for solutions containing quenchers was calculated by means of (1) on the ground of a simplified model of a luminescent centre. The following simplifying assumptions were made: 1) the quenching probability (per unit time) is assumed to be different from zero only for quenching by quenchers situated within a certain active sphere of the luminescent molecule, 2) the volume of the active

sphere is so defined that the quenching probability by one quencher in the active sphere is equal to that of deactivation of the excited luminescent molecule in a solution free of quenchers, 3) the quenching probability by k quenchers is k times that of one quencher, 4) mutual diffusion of quenchers and luminescent molecules is neglected (the last assumption is fulfilled rigorously in cases of rigid solutions). Thus, neither the influence of mutual orientations of molecular axes of quenchers and luminescent molecules was taken into account, nor any specific form assumed of the dependence of quenching probability on the distance of quenchers from luminescent molecules.

There follows from the above assumptions

$$\frac{\eta_k}{\eta_0} = \frac{1}{1+k} \tag{3}$$

for a centre with k quenchers, η_0 being the yield of centres with no quenchers ($k = 0$).

By applying for P_k the Smoluchowski distribution [5], [6]

$$P_k = e^{-\nu} \frac{\nu^k}{k!}, \tag{4}$$

the following expression was obtained from (1) and (3):

$$\frac{\bar{\eta}}{\eta_0} = e^{-\nu} \sum_{k=0}^{\infty} \frac{\nu^k}{(1+k)!} = \frac{1-e^{-\nu}}{\nu}, \tag{5}$$

where $\nu = nv$ is the mean value of the number of quenchers in active spheres of volume v cm³ at the concentration n cm^{-3} of quenchers.

The expression (5) was shown to agree well with the experimental results of Bowen and Brocklehurst [1] and with those of Förster [2] (cf. [4]), if v is appropriately chosen (v is the only constant in (5) to be fitted to the experimental results).

A better approximation should result by using the "shell model" of a luminescent centre [3]. In that model a short range order of solvent molecules surrounding the luminescent molecule is assumed, quenchers being imbedded in different shells constituted of solvent molecules, their distances R thus assuming discrete values only. This picture seems to correspond closer to real conditions, than those in which R is assumed to vary continuously. A generalized Smoluchowski distribution

$$P(k_1, k_2 \ldots k_z) = e^{-n \sum_{l=1}^{z} v_l} \prod_{l=1}^{z} \frac{(nv_l)^{k_l}}{k_l!} \tag{6}$$

may be applied in this case, z denoting the assumed number of shells, v the volume of the shell l and k_l the population of the shell l by quenchers.

By means of (6)

$$\bar{\eta} = \sum_{k_1=0}^{\infty} \dots \sum_{k_z=0}^{\infty} P(k_1 \dots k_z)\, \eta(k_1 \dots k_z) \tag{7}$$

can be calculated, if $\eta(k_1 \dots k_z)$ is known. Such calculations are not made in this paper. An improved model of centre with an active sphere is rather used, in which the dependence of quenching probability on distance R as well as quenching by quenchers outside the active sphere is taken into account.

Quenching by energy transfer

According to Förster [2] the quenching probability (per unit time) W is inversely proportional to the sixth power of the distance R of the quencher from the luminescent molecule in cases, in which quenching is due to the excitation energy transfer from luminescent molecules to quenchers:

$$W = \frac{1}{\tau}\left(\frac{R_0}{R}\right)^6, \tag{8}$$

where τ is the mean life of the excited state of luminescent molecules in a solution free of quenchers, and R_0 the distance of quencher at which $W = \frac{1}{\tau}$.

On assumption of additivity of W the relative yield (η_0 denoting the yield in a solution with no quenchers) for a given configuration of z quenchers is

$$\frac{\eta(R_1 \dots R_z)}{\eta_0} = \frac{\frac{1}{\tau}}{\frac{1}{\tau} + \frac{R_0^6}{\tau}\sum_{l=1}^{z} R_l^{-6}} = \frac{1}{1 + R_0^6 \sum_{l=1}^{z} R_l^{-6}}. \tag{9}$$

The observed relative yield of the whole solution, i. e. the mean value of the expression (9), could be calculated by means of (2), or, even better, by means of (6) and (7). We shall, however, proceed in a simpler, although rather only approximate, way. We again use the notion of the active sphere and define its radius R' by condition that the mean value of the relative yield of centres containing one quencher be $\eta_1/\eta_0 = \frac{1}{2}$ provided the quenching by outer quenchers, i. e. by quenchers situated at $R > R'$, is neglected.

For the case of quenching by one quencher, (9) reduces to

$$\frac{\eta_1(R)}{\eta_0} = \frac{1}{1 + \left(\frac{R_0}{R}\right)^6}. \tag{10}$$

The probability distribution of R being $\frac{3}{R'^3}R^2dR$, we have from Eqs. (2) and (10)

$$\frac{\bar{\eta}_1}{\eta_0} = \frac{3}{R'^3}\int_0^{R'} \frac{R^2}{1+\left(\frac{R_0}{R}\right)^6} dR = 1 - \frac{\operatorname{arctg}\left(\frac{R'}{R_0}\right)^3}{\left(\frac{R'}{R_0}\right)^3}. \tag{11}$$

Imposing on (11) the above condition $\bar{\eta}_1/\eta_0 = \frac{1}{2}$, we obtain for the radius of active sphere

$$R' = 1.327\, R_0\,. \tag{12}$$

After the radius of the active sphere is thus established, we proceed to approximate calculations of the contribution to the quenching brought in by outer quenchers (having $R > R'$), and then to calculations of the relative yield of centres containing $k \neq 0$ inner quenchers.

Let us consider a centre with no inner quenchers, and assume the spatial distribution of outer quenchers to be uniform. The last assumption introduces an error, which may be considered to be rather small for low concentrations of quenchers. Let there be N outer quenchers in a concentric with v large spherical volume V, whose radius is R''. The sum $\sum_{l=1}^{N} R_l^{-6}$ in denominator of Eq. (9) for N quenchers distributed uniformly in V is

$$\sum_{l=1}^{N} R_l^{-6} = \frac{4\pi N}{V}\int_{R'}^{R''} \frac{R^2}{R^6} dR = \frac{4\pi n}{3}\left(\frac{1}{R'^3} - \frac{1}{R''^3}\right) \approx \frac{4\pi n}{3R'^3}. \tag{13}$$

Since

$$n = \frac{N}{V} = \frac{\nu}{v} = \frac{\nu}{\frac{4}{3}\pi R'^3},$$

$$\sum_{l=1}^{N} R_l^{-6} = \frac{\nu}{R'^6}. \tag{14}$$

Putting (14) into (9) we obtain for the relative yield η_0'/η_0 of a centre without inner quenchers:

$$\frac{\eta_0'}{\eta_0} = \frac{1}{1+\left(\frac{R_0}{R'}\right)^6 \cdot \nu}. \tag{15}$$

The calculations of mean relative yields $\bar{\eta}_k/\eta_0$ of centres with $k \neq 0$ inner quenchers are made easier, if advantage is taken of the following identities:

(16) $$\frac{\eta_k}{\eta_0} \equiv \frac{\eta_k}{\eta_{k-1}} \frac{\eta_{k-1}}{\eta_0}$$

and

(16a) $$\frac{\bar{\eta}_k}{\eta_0} \equiv \frac{\bar{\eta}_k}{\bar{\eta}_{k-1}} \frac{\bar{\eta}_{k-1}}{\eta_0}.$$

The problem of computation of $\bar{\eta}_k/\eta_0$ is reduced by using (16) and (16a) to successive calculations of $\bar{\eta}_k/\bar{\eta}_{k-1}$ for $k = 1, 2, \ldots$

If follows from (9)

(17) $$\frac{\eta_k}{\eta_{k-1}} = \frac{1 + R_0^6 \sum_{l=1}^{k-1} R_l^{-6}}{1 + R_0^6 \sum_{l=1}^{k} R_l^{-6}} = \frac{1}{1 + \eta_{k-1} \left(\frac{R_0}{R_k}\right)^6}.$$

We introduce here a further approximation by putting

(18) $$\frac{\eta_k}{\bar{\eta}_{k-1}} = \frac{1}{1 + \bar{\eta}_{k-1} \left(\frac{R_0}{R_k}\right)^6} = \frac{1}{1 + \left(\frac{\varrho_{k-1}}{R_k}\right)^6},$$

where $\varrho_{k-1} = R_0 \cdot \bar{\eta}_{k-1}^{-6}$. The mean value of (18) is then calculated in a way similar to that used in (11):

(19) $$\frac{\eta_k}{\bar{\eta}_{k-1}} = \frac{3}{R'^3} \int_0^{R'} \frac{R^2}{1 + \left(\frac{\varrho_{k-1}}{R}\right)^6} dR = 1 - \frac{\operatorname{arctg}\left(\frac{R'}{\varrho_{k-1}}\right)^3}{\left(\frac{R'}{\varrho_{k-1}}\right)^3}.$$

By means of identity (16a) and Eq. (19) the mean relative yields are calculated for various k and ν, and then by means of Eqs. (1) and (4) the relative yield $\bar{\eta}/\eta_0$ of the whole solution as a function of ν (i. e. that of concentration) obtained. It is to be noted that also in this new version of the theory the only constant to be fitted to the experimental results is the volume v of the active sphere. Fig. 1 shows the relative yield thus calculated as a function of logarithm of ν together with that resulting from the former simpler form of the theory (Eq. (5)). The two curves appear to be shifted with respect to one another (which means that the volume of the active sphere must be chosen differently for the two curves for fitting them to the same experimental results *)), and their slope is somewhat different.

*) The best fit of the new curve to the experimental results of Förster [2] for trypaflavine in methanol quenched by rhodamine B is obtained for $R' = 70.6$ Å ($R_0 = 53.4$ Å), compared with $R' = 78$ Å ($R_0 = 58.8$ Å) fitting to Eq. (5). The experimental value of R_0 calculated by Förster on the ground of his theory is $R_0 = 58$ Å, while $R_0 = 68$ Å is quoted by him as a value expected on the ground of theoretical considerations.

Fig. 2 shows the relative yield as a function of v resulting from the present calculations together with experimental points for trypaflavine in methanol quenched by rhodamine B due to Förster [2]. The deviation

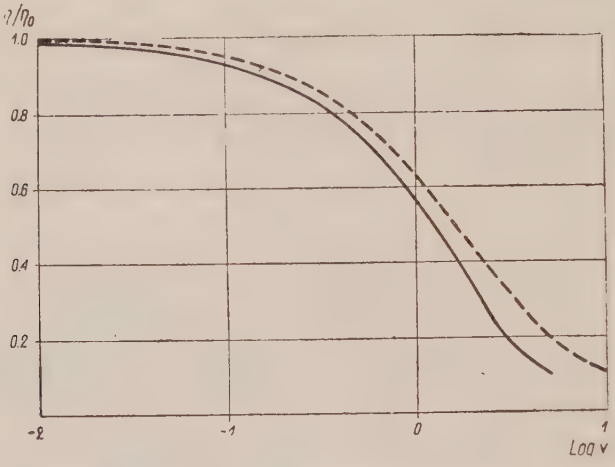

Fig. 1. $\bar{\eta}/\eta_0$ *versus* logarithm of v; —— present theory; ---- earlier theory

of experimental points from the calculated curve seems to lie everywhere within the limits of experimental errors, with, possibly, one exception of the point at the highest concentration of quenchers ($v = 4.42$).

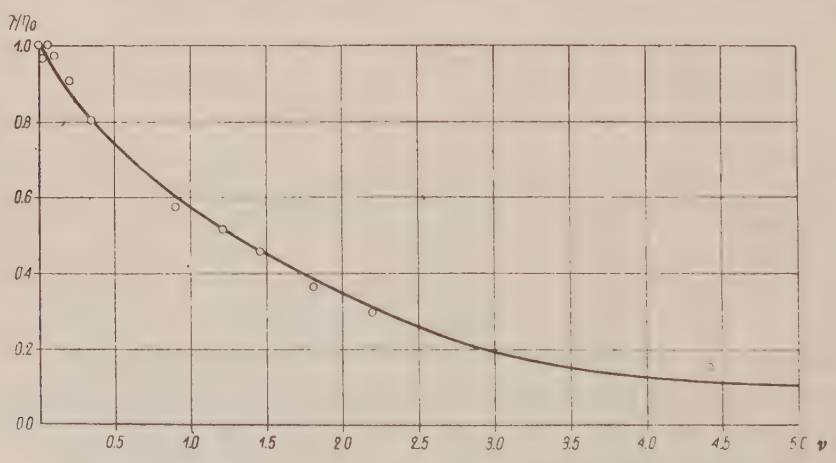

Fig. 2. $\bar{\eta}/\eta_0$ *versus* v. Calculated curve together with experimental points of Förster for trypaflavine in methanol quenched by rhodamine B, $R' = 70.6$ Å ($R_0 = 53.4$ Å) assumed

The new curve seems to fit still better to the experimental results, than that given by Eq. (5), resulting from the former simpler form of the theory (see [4], Fig. 1). This fact, however, should not be considered

as a final justification of Eq. (8), since the slope of the curve is not very sensitive on the specific form of the dependence of W on R.

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On the Dependence of Emission Anisotropy of Fluorescence on the Concentration of Luminescent Molecules in Plexiglass Luminophors

by

A. KAWSKI

Presented by A. JABŁOŃSKI on August 1, 1958

Investigations by the present author [1] concerning the dependence of the degree of polarization on the concentration of anthracene and fluorescein in methyl polymethacrylate (plexiglass) proved anthracene solutions in plexiglass to exhibit concentrational depolarization. When the concentration was not too high, the decrease in the degree of polarization with the rise of anthracene concentration was found to be in agreement with Jabłoński's theory of self-depolarization [2]. However, rigid solutions of fluorescein ($C_{20}H_{12}O_5$, molecular weight 332.30) showed no decrease in polarization over a considerable range of rising concentrations (from 3×10^{-6} to 6×10^{-3} g/cm^3), for wave-lengths of $\lambda = 366$ mμ, $\lambda \approx 436$ mμ and $\lambda \approx 490$ mμ.

The present investigation deals with the dependence of the emission anisotropy (degree of polarization) on the concentration of "yellowish water-soluble eosin" (microscopic dye, molecular weight 687.026, Fig. 1)

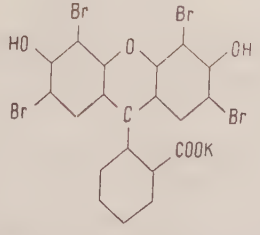

Fig. 1. Structural formula of "water soluble yellowish eosin"

and "rhodamin *B*" (molecular weight 442.53) in methyl polymethacrylate. The rhodamin *B* was purified by crystallization. Rigid plexiglass solutions of these dyes were prepared by a method described in an earlier paper by the present author [1].

Instead of the degree of polarization, it is more convenient to employ the emission anisotropy r as introduced by Jabłoński [3]. For plane polarized exciting light

$$r_p = \frac{I_\parallel - I_\perp}{I_\parallel + 2I_\perp} = \frac{2P_p}{3 - P_p} \tag{1}$$

and for natural light

$$r_n = \frac{I_\parallel - I_\perp}{2I_\parallel + I_\perp} = \frac{2P_n}{3 + P_n}, \tag{2}$$

where $I_\parallel$ and $I_\perp$ are the components of the emitted intensity parallel and perpendicular to the direction of vibration of the electric vector of the exciting light, and P_p, P_n denote the degrees of polarization as obtained with plane polarized and natural exciting light, respectively. In order to eliminate the effect of secondary fluorescence, measurements of emission anisotropy as dependent on concentration were made with samples in the shape of plates of suitable thickness. The emission anisotropy was measured by means of a Savart polariscope with Arago compensator and a (Hilger) spectograph [4], [5], [1]. On passing a Zeiss mirror monochromator, the exciting light was polarized with a polarizing Glan prism. In the present investigation, the luminophors were excited within the region of the absorption maximum of yellowish eosin and rhodamin *B*. The interference picture produced by the Savart polariscope was projected into the slit of the spectograph (in this case, a magnifying glass was used for observation [6]). By means of an Arago compensator, the interference fringes visible on the fluorescence band were made to vanish. Readings of the angle a corresponding to compensation by the set of glass plates were taken, whence the emission anisotropy was computed from the following formula (in place of the quantities P_p and P_n in Eqs. (1) and (2), substitution is effected in the general formula yielding the degree of polarization for k glass plates [7], [8]):

$$r_{p.n} = \frac{2\left(1 - \frac{1}{n^2}\right)^2 \sin^2 a}{3\left(1 + \frac{1}{n^2}\right)\left[2 - \left(1 + \frac{1}{n^2}\right)\sin^2 a\right] \mp \left(1 - \frac{1}{n^2}\right)^2 \sin^2 a + 12\left(\frac{1-k}{k}\right)\frac{\cos a}{n^2}\sqrt{n^2 - \sin^2 a}} \tag{3}$$

the sign „$-$" is taken in the case of exciting light plane polarized, and „$+$" in that of natural light where the alternative „$\pm$" appears. The author used an Arago compensator consisting of four glass plates ($k = 4$).

The Vavilov-Levshin formula [9] relating to the degree of polarization of fluorescence P_n obtained with natural exciting light and P_p resulting from plane polarized exciting light assumes the following particularly simple form when expressed in terms of emission anisotropy:

$$\frac{r_p}{r_n} = 2. \tag{4}$$

It will be seen from Table I that the measured values of r_p and r_n fulfill relation (4) satisfactorily. The values of the fundamental degree of polarization of the dyes investigated in methyl polymethacrylate solution, as obtained by the present author and by E. Laffitte [5] will also be found in Table I. In the case of rhodamin *B* in plexiglass, at a concentration of $C \approx 10^{-5}$ g/cm³, a value of only $P_0 = 0.415$ was obtained, whereas E. Laffitte found $P_0 = 0.46$, corresponding to $C \approx 10^{-6}$ g/cm³. In the present investigation it was not possible to measure polarization at concentrations below 10^{-5} g/cm³ because of the weak emission of the rhodamin *B* luminophor.

TABLE I

Substance fluorescent in plexiglass	r_p	r_n	$\frac{r_p}{r_n}$	Fundamental degree of polarization P_0	
				Author	Laffitte
Yellowish water soluble eosin	0.353	0.179	1.97	0.45	—
Eosin $C_{20}H_8O_5Br_4$ (tetrabromofluorescein)	0.349	0.179	1.95	0.445	—
Rhodamin *B*	0.321	0.162	1.98	0.415	0.46
Fluorescein $C_{20}H_{12}O_5$	0.353	0.172	2.05	0.45	0.45

The effect of the concentration of the dye on emission anisotropy (degree of polarization) was investigated within the range of low concentrations of yellowish eosin and rhodamin *B* in plexiglass solutions, since it was practically impossible to obtain higher concentrations of the dyes. For yellowish eosin, the concentrations ranged from $2 \cdot 10^{-6}$

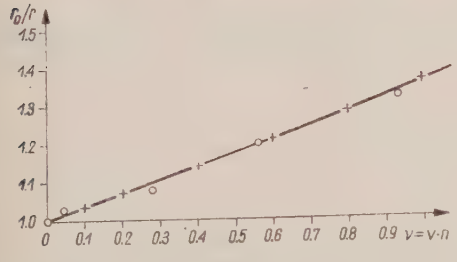

Fig. 2

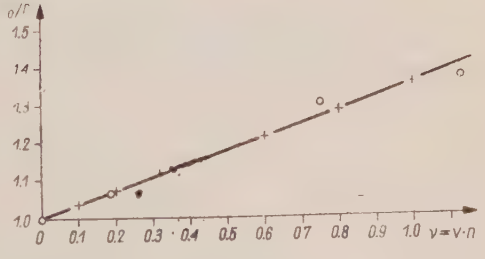

Fig. 3

Fig. 2. r_0/r *versus* $\nu = v \cdot n$ for yellowish eosin
(n — number of luminescent molecules in 1 cm³)
+ — points calculated by means of Eq. (8)
○ — experimental results.

Fig. 3. r_0/r *versus* $\nu = v \cdot n$ for rhodamin B
+ — points calculated by means of Eq. (8)
○ — experimental results

to $4.3 \cdot 10^{-4}$ g/cm^3, whilst for rhodamin B the corresponding values were $1.1 \cdot 10^{-5}$ and $1.5 \cdot 10^{-4}$ g/cm^3. The luminophors exhibited a decrease of emission anisotropy with the rise of concentration of the dyes (Figs. 2 and 3).

Jabłoński's most recent theory of self-depolarization (1955), [2], where, as in other theories, the cause of self-depolarization of photoluminescence is considered to consist in the migration of excitation energy, differs from other theories in that it takes into account the fluctuation of the concentration of luminescent molecules and the re-emigration of excitation energy. Transmission of the excitation energy takes place within the so-called "active sphere". The expression given by Jabłoński [11], which describes the dependence of the degree of polarization on the concentration in the absence of quenching by molecules of the same and of another kind, is of the following shape:

$$P = 3P_0 \frac{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \cdot \frac{2}{k+1}}{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \left[(3 - P_0) + \frac{2P_0}{k+1} \right]}, \tag{5}$$

where $\nu = v \cdot n$, n denotes the number of luminescent molecules in 1 cm^3, v — the volume of the active sphere, and P_0 — the value of P for $\nu = 0$.

Eq. (5) may be rewritten as follows [10]:

$$\frac{1}{P} = \frac{1}{P_0} + \left(\frac{1}{P_0} - \frac{1}{3} \right) \left[\frac{\nu^2}{2(\nu - 1 + e^{-\nu})} - 1 \right]. \tag{6}$$

If $\nu \ll 1$, Eq. (6) takes the form:

$$\frac{1}{P} \approx \frac{1}{P_0} + \left(\frac{1}{P_0} - \frac{1}{3} \right) \frac{1}{3} \nu . \tag{7}$$

In the case considered, Eqs. (6) and (7) expressed in the emission anisotropy take the form:

$$\frac{r_0}{r} = \frac{\nu^2}{2(\nu - 1 + e^{-\nu})} \tag{8}$$

and, for $\nu \ll 1$,

$$\frac{r_0}{r} \approx 1 + \frac{1}{3} \nu , \tag{9}$$

where r_0 is the emission anisotropy for $\nu = 0$ (the fundamental emission anisotropy).

In Figs. 2 and 3, the theoretical dependence of r_0/r on $\nu = v \cdot n$ resulting from Eq. (8) is shown (ν denotes the concentration multiplied by the volume of the "active sphere") and compared with measurements made with solutions of yellowish eosin and rhodamin B in plexiglass,

for active spheres of radii $R = 84$ Å and $R = 111$ Å, respectively. It will be seen from Figs. 2 and 3 that, within the range of (low) concentrations investigated, the experimental results are in excellent agreement with the theory put forth by Jabłoński.

The author wishes to express his thanks to Professor A. Jabłoński for his numerous and valuable hints, and his indebtedness to Professor W. Mościcki, Head of the Department of Physics, where the present investigation was effected, for his interest in this work. He is indebted to Mr. J. Głowacki and Mrs. B. Polacka for their co-operation in preparing the plexiglass luminophors.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 10

К. СЕКЛЮЦКИЙ, ТОПОЛОГИЧЕСКИЕ СВОЙСТВА МНОЖЕСТВ, НА КОТОРЫХ ИМЕЮТСЯ СИСТЕМЫ ЧЕБЫШЕВА . . . . . стр. 603—606

Через Z обозначим замкнутое и ограниченное подмножество эвклидова пространства или в общем компактное топологическое пространство. Через $C(Z)$ обозначим множество всех действительных и непрерывных функций на Z.

Система функций $\varphi_1, \varphi_2, \ldots, \varphi_n \in C(Z)$ называется системой Чебышева, если для любых чисел c_k $(k = 1, 2, \ldots, n)$ таких, что $\sum_{k=1}^{n} |c_k| > 0$, уравнение $\sum_{k=1}^{n} c_k \varphi_k(p) = 0$ имеет, по крайней мере, $n-1$ корней. Значение понятия системы Чебышева для теории аппроксимации выясняет известная теорема Хаара [1].

Возникает вопрос, при каких условиях на Z существует система Чебышева, состоящая из n функций из $C(Z)$ $(n>1)$. С. Мазур высказал предположение, что это имеет место только в таком случае, когда пространство Z гомеоморфно окружности или некоторому подмножеству отрезка.

Целью работы является доказать это предположение в виде следующей теоремы: *если на пространстве Z существует систета Чебышева, состоящая из n функций, то в случае четного n, пространство Z гомеоморфно подмножеству отрезка, а в случае нечетного $n\,(n>1)$ — Z гомеоморфно окружности или подмножеству отрезка.*

Л. ДУБИКАЙТИС, О ПОРЯДКЕ ТОЧЕК И ГИПЕРПЛОСКОСТЕЙ В n-МЕРНОЙ ПРОЕКТИВНОЙ ГЕОМЕТРИИ . . . . . стр. 607—610

В настоящей работе автор вводит $(n+3)$-местное отношение: $\{A_1, \ldots, A_{n+3}\}$, определенное в множестве точек n-мерного проективного пространства. Это отношение является обобщением отношения разделенности пар точек на прямой.

В работе ставится проблема замены в аксиоматике n-мерной проективной геометрии понятия разделенности пар точек на прямой понятием отношения $\{a_1, \ldots, a_{n+3}\}$, определенного в множестве гиперплоскостей.

С. МИКОЛАЙСКАЯ и Т. ВАЖЕВСКИЙ, ОПЕРАТОРЫ, ТРАНСФОРМИРУЮЩИЕ СХОДЯЩИЕСЯ РЯДЫ В СХОДЯЩИЕСЯ РЯДЫ стр. 615—618

Гипотеза Н. *Даются два Банаховы линейные пространства X и Y с однородной нормой и абстрактная функция — оператор $f(x)$, определенный в пространстве X; его значения принадлежат к пространству Y.*

Предметом работы является соотношение между тремя следующими свойствами функции $f(x)$.

Свойство A состоит в том, что функция $f(x)$ превращает всякий сходящийся ряд в сходящийся ряд, т. е. что из сходимости ряда $\sum_{v=1}^{\infty} x_v$ следует сходимость ряда $\sum_{v=1}^{\infty} f(x_v)$.

Свойство B состоит в том, что 1) f — непрерывна для $x=0$ и что 2) имеется $r(r>0)$ такое, что $f(a)+f(b)=f(a+b)$, если $a \in X$, $b \in X$, $|a| \leqslant r$, $|b| \leqslant r$.

Свойство C состоит в том, что имеется линейный оператор $F(x)$ и число $r(0<r<\infty)$ такое, что $f(x)=F(x)$ для $|x| \leqslant r$.

Теорема 1. *Принимая гипотезу* Н, *свойства A, B, C равносильны.*

Замечание. *Принимая гипотезу* Н, *оказывается, что условием необходимым и достаточным для того, чтобы оператор $y=f(x)$ превращал всякий абсолютно сходящийся ряд в абсолютно сходящийся ряд, является*

$$f(0)=0, \quad \limsup_{x\to\infty} \frac{|f(x)|}{|x|} < +\infty .$$

С. Г. РОГОЗИНСКИЙ, АННИГИЛЯЦИЯ ПАРЫ ЭЛЕКТРОН-ПОЗИТРОН С ПРОИЗВОЛЬНОЙ ПОЛЯРИЗАЦИЕЙ . . . . . стр. 619—626

В работе введено выражение на эффективное сечение в произвольной системе координат для двуквантовой аннигиляции пары электрон-позитрон с произвольной поляризацией в случае если поляризация фотонов не подлежит наблюдению. Это является обобщением выражения на эффективное сечение в системе центра массы для аннигиляции частиц поляризованных продольно и поперечно, вычисленного Л. А. Пэйджем [3].

На основании полученных результатов продискутирован некоторый метод измерения поляризации позитронов, происходящих из распада β^+. Этот метод, состоящий в наблюдении аннигиляции позитронов на электронах в магнетизированной ферромагнитной среде, применяли С. С. Ганна и Р. С. Прэстон [1], [2].

Измерения поляризации играют важную роль в проблемах, связанных с несохранением четности.

А. ТРАУТМАН, О ГРАВИТАЦИОННОМ ЛУЧИСТОМ ТРЕНИИ . . . . . стр. 627—633

Рассматриваются лучистые эффекты, связанные со свободным движением тяжелых тел при применении метода Эйнштейна-Инфельда-Гоффманна. Оказы-

вается, что n-тый член разложения метрического тензора $g_{\mu\nu}$ в ряд относительно $1/c$ ведет себя в общем, по крайней мере, как r^{n-5} $(n \geqslant 4)$. Нечетные члены в разложении g_{00} и g_{ik}, а также четные члены в разложении g_{0k} называем лучистыми. Дается конечное и достаточное условие для того, чтобы комплекс полей $(\underset{n}{g_{ik}}, \underset{n+1}{g_{0k}}, \underset{n+2}{g_{00}})$ можно было устранить путем преобразования системы координат. Для $n = 5$ получаем первый комплекс членов, не аннулирующийся никаким преобразованием системы координат. Выражения $\underset{5}{g_{ik}}$ и $\underset{6}{g_{0k}}$ можем получить непосредственно из линеаризованной теории, а $\underset{7}{g_{00}}$ решая соответственное уравнение поля. Знание этих функций является достаточным для нахождения лучистых исправлений 9-го порядка для уравнения движения (ньютоновские уравнения — 4-го порядка).

В случае двух тел с равными массами m, движущихся в нелучистом приближении равномерно по окружности с радиусом R, уравнения движения 9-го порядка имеют вид $\ddot{x} + 2a\dot{x} + \omega^2 x = 0$, где $a = 3m^3/20R^4 > 0$.

И. ДОМБРОВСКИЙ, ЭНЕРГИЯ СВЯЗИ ЯДРА ^{16}O ПРИ ПРИМЕНЕНИИ МЕТОДА „*cluster*"-РАЗЛОЖЕНИЯ . . . . . стр. 635—643

В работе вычислена энергия связи и размеры ядра ^{16}O при помощи вариационного метода с применением пробной функции в форме, учитывающей корреляцию между нуклонами.

Принимаются следующие предположения: бесконечное ядро, силы действующие между двумя нуклонами описаны взаимодействием К. Юкавы с отталкивающим остовом и с обменом Сербера.

Для исчисления ожидаемого значения энергии был применен метод „*cluster*" разложения. В разложении относительно $\dfrac{\Omega_c}{\Omega}$ (Ω_c — полный корреляционный объем, Ω — объем системы) учитывались только два члена, параметры функции определялись из нулевого приближения.

Полученные оценки свидетельствуют, о том, что из принятого потенциала можно получить хорошую энергию связывания и плотность ядра ^{16}O для $0{,}2f < a < 0{,}6f$ (a — радиус отталкивающего остова), параметр корреляционной функции консистентен с оболочной моделью. В приписке выведено уравнение для корреляционной функции.

В. ТУЛЬЧИЕВ, ДВИЖЕНИЕ ВРАЩАЮЩИХСЯ ТЕЛ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ . . . . . стр. 645—651

В работе приводятся уравнения движения вращающихся тел в посленьютоновском приближении, а также проводится интегрирование этих уравнений, получая выражения на перемещение периастрона и прецессию узлов.

Р. БАУЭР, А. БОНЧИНСКИЙ и М. ЧАЙКОВСКИЙ, МИГРАЦИЯ ЭНЕРГИИ ВОЗБУЖДЕНИЯ ИЗ МОЛЕКУЛ КРАСИТЕЛЯ В МЕТАСТАБИЛЬНОМ СОСТОЯНИИ . . . . . стр. 653—658

Исследована зависимость анизотропии эмиссии света флуоресценции и фосфоресценции от концентрации молекул красителя для акридинового желтого в борной кислоте. Сопоставлены результаты исследований с теорией А. Яблонского. Констатирована миграция энергии из метастабильного уровня молекулы.

А. ТРАМЕР, СПЕКТР КОМБИНАЦИОННОГО РАССЕЯНИЯ МОНОКРИСТАЛЛА ТРЕХХЛОРИСТОЙ СУРЬМЫ . . . . . стр. 659—661

Спектр состоит из нескольких малых частот, которые отвечают колебаниям решетки и из шести линий, отвечающих внутримолекулярным колебаниям.

Сравнение со спектром жидкости указывает на расщепление вырожденных частот в кристалле.

Снятие вырождения объясняется низшей симметрией молекулы в кристалле.

А. ЯБЛОНСКИЙ, МИГРАЦИОННОЕ ТУШЕНИЕ ФОТОЛЮМИНЕСЦЕНЦИИ РАСТВОРОВ . . . . . стр. 663—669

Уточнена, предложенная автором в предыдущих работах, теория тушения фотолюминесценции. Вычислена зависимость выхода флуоресценции от концентрации тушителей для случая миграционного тушения, в котором вероятность миграции обратно пропорциональна шестой степени расстояния тушителя от люминесцентной молекулы.

Теоретическая кривая хорошо согласуется с экспериментальными результатами Ферстера.

А. КАВСКИЙ, ЗАВИСИМОСТЬ АНИЗОТРОПИИ ЭМИССИИ ФЛУОРЕСЦЕНЦИИ ОТ КОНЦЕНТРАЦИИ ЛЮМИНЕСЦИРУЮЩИХ МОЛЕКУЛ В ПЛЕКСИГЛАССОВЫХ ЛЮМИНОФОРАХ . . . . . стр. 671—675

Исследована зависимость анизотропии эмиссии (поляризации) от концентрации для желтоватого эозина (от $2 \cdot 10^{-6}$ до $4{,}3 \cdot 10^{-4}$ *г/см*^{3}), а также родамина *B* (от $1{,}1 \cdot 10^{-5}$ до $1{,}5 \cdot 10^{-4}$ *г/см*^{3}) в полиметакриловом метиле (плексиглассе).

Сопоставление полученных экспериментальных результатов (снижение анизотропии эмиссии по мере роста концентрации красителей) с теоретическим выражением А. Яблонского, описывающим зависимость анизотропии эмиссии (а также степени поляризации) от концентрации, в случае если не имеет место тушение собственными и чужими молекулами, указывает в случае вышеупомянутого предела концентрации на весьма хорошую согласованность эксперимента с теорией самодеполяризации Яблонского. Измерены также основные степени поляризации для двух различных эозинов, родамина *B* и флуоресцеина в плексиглассе.

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Cena zł 20.—